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Margaret Daly Davis

Piero della Francesca's Mathematical Treatises

The «Trattato d'abaco» and «Libellus de quinque corporibus regularibus»

Longo Editore - Ravenna

Questo volume è uno studio dei rapporti fra la teoria matematica e la pratica artistica nel Rinascimento. L'autrice esamina il caso specifico di Piero, l'artista oggi universalmente ammirato per la perfezione geometrica delle sue opere. Benché il trattato sulla prospettiva di Piero venga discusso tutte le volte che la prospettiva è all'ordine del giorno, le sue opere di matematica pura, il *Trattato d'abaco* e il *Libellus de quinque corporibus regularibus*, sono state trascurate dagli storici dell'arte. E', però, attraverso questi due trattati « minori » che la capacità matematica di Piero, celeberrimo al suo tempo, viene di nuovo messa a fuoco.
L'autrice conduce un'indagine particolareggiata su questi trattati, tracciandone le fonti nella matematica classica e tardo-medievale e studiando lo sviluppo interno della matematica pieriana, come pure i rapporti con il trattato sulla prospettiva dello stesso Piero. Di seguito alla esposizione del contenuto delle due opere, il lavoro analizza le implicazioni delle idee di Piero e la loro diffusione nella teoria pratica delle arti figurative e dell'architettura.

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In copertina: ritratto di Luca Pacioli, *Summa de arithmetica*, fol. 111v.

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1.

Margaret Daly Davis

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Margaret Daly Davis

Piero della Francesca's Mathematical Treatises

The «Trattato d'abaco» and «Libellus de quinque corporibus regularibus»

LONGO EDITORE - RAVENNA

RIASSUNTO

1.

Artisti e matematici: studi aritmetici, geometrici e prospettici

In tutta la critica moderna sull'arte di Piero della Francesca, l'aspetto geometrico della sua opera, quell'astratta precisione quasi matematica della sua pittura, viene sempre presa in considerazione. Per il Longhi è « *come se ogni cosa dovesse diventare uno dei cinque corpi regolari per servire veramente alla costruzione prospettica, che è costruzione centrale assestata su piani convergenti ad un asse ideale, in quanto ci vuol far risentire lo spazio come volume regolare per mezzo dell'appostamento in esso di altri volumi regolari...* » E già il Vasari ricorda che nelle cose di geometria e prospettiva Piero « *non fu inferiore a niuno ne' tempi suoi, né forse che sia stato in altri tempi giammai; come ne dimostrano tutte l'opere sue piene di prospettive* ».

Quindi è strano che i due trattati matematici di Piero, il *Trattato d'abaco* e il *Libellus de quinque corporibus regularibus,* non siano stati studiati a fondo, malgrado vengano sempre menzionati come citazioni d'obbligo nel gran numero di pubblicazioni che riguardano la sua produzione artistica. La situazione è forse comprensibile poiché, a differenza della sua *Prospectiva pingendi,* i trattati minori di Piero, appunto quelli matematici, non sembrano a prima vista avere una relazione diretta e trasparente con le arti figurative. Questo lavoro mira a mettere in evidenza alcuni di tali rapporti e parte da un at-

tento esame dell'*Abaco* e del *Libellus,* cercando di chiarirne i contenuti, le fonti e il significato storico, anche nello stretto senso matematico.

Al suo tempo gl'interessi matematici di Piero erano in sé del tutto normali per un artista, pur ammettendo che nel suo caso erano altamente sviluppati. Il Rinascimento visse una fiorente collaborazione fra le belle arti e le scienze matematiche di aritmetica e geometria, una connessione già evidente nel Trecento e nel Quattrocento. I molteplici rapporti fra i matematici e gli artisti del tempo vengono brevemente esaminati in alcuni casi tipici, specialmente nel campo della trattatistica, ad esempio, *I commentari* del Ghiberti, l'amicizia fra il Brunelleschi e il matematico Paolo Toscanelli, le relazioni di Luca Pacioli con Leonardo da Vinci, ed alcune situazioni venete. Anche se l'attività matematica di Piero va inserita nel contesto della viva e tradizionale sodalità fra artisti e matematici, essa fa parte del più generale risveglio dell'antichità classica, trovando un fondamento non soltanto in Euclide, ma anche in Archimede, Tolomeo, Vitruvio e Plinio.

Il primo trattato pieriano, il *Trattato dell'abaco* nella Biblioteca Laurenziana (Ashb. 280) appartiene a un genere di testo matematico che, in principio, serviva ad insegnare la matematica mercantile. Con il passare del tempo fu aggiunta una sezione geometrica, e così gli abbachi vennero utilizzati come uno strumento più generale nell'insegnamento matematico, anche nel campo artistico-tecnico. Con un'impostazione molto meno pratica del libro dell'abbaco comune, l'*Abaco* di Piero tratta la materia in un modo quasi puramente matematico, e passa oltre i confini dell'abbaco normale per considerare problemi sofisticati di geometria stereometrica, aprendo così la strada al *Libellus* sui corpi regolari.

A differenza dell'*Abaco,* il *Libellus de quinque corporibus regularibus* non rientrava in un genere di trattato già stabilito. Per la prima volta i cinque corpi regolari — il tetraedro, il cubo, l'ottaedro, l'icasoedro, e il dodecaedro — vengono trat-

tati in un libro affine agl'interessi degli artisti, e in seguito il tema verrà riproposto in manuali e trattati più prettamente artistici. Il trattato di Piero si ispira rigorosamente agli *Elementi* di Euclide, mentre le implicazioni cosmologiche dei corpi regolari di Platone, il quale attribuisce ad essi un profondo significato filosofico, vengono tralasciate.

Il *Libellus* si divide in quattro parti. La prima si occupa di figure geometriche bi-dimensionali; la seconda, dell'iscrizione di corpi stereometrici nella sfera; la terza, dell'iscrizione di un corpo regolare in un altro; e l'ultima, di corpi geometrici irregolari.

2.

Il « Trattato d'Abaco »: algebra e geometria

1. *La provenienza del manoscritto dell'« Abaco » e il suo contenuto algebrico*

L'*Abaco* di Piero, un manuale di matematica mercantile, fu commissionato da un Pichi di Borgo Sansepolcro, un mercante, per quanto sembra, che aveva un interesse fuori del comune per questioni matematiche. Identificato per primo da Girolamo Mancini nel 1917 come opera di Piero, il manoscritto dell'*Abaco* si trovava nella collezione Ashburnham di Londra, in gran parte proveniente dalla collezione di Guglielmo Libri, bibliotecario, storico della matematica e appassionato collezionista di libri e manoscritti antichi. Difatti l'*Abaco* laurenziano faceva parte della collezione Libri (verosimilmente fu portato via da Sansepolcro soltanto nell'800, forse dallo stesso Libri, una figura dai tratti oscuri). La prova di tale provenienza è semplice. Nel 1840 a Parigi il Libri stampò un lungo estratto dell'*Abaco* (che allora gli apparteneva) nel suo *Histoire de la mathématique en Italie,* identificandolo erroneamente con un'opera toscana del '300. Più acuta l'analisi del Libri sul contenuto matematico del testo, in cui troviamo il tentativo di risolvere equazioni algebriche superiori al secondo grado, una soluzione realizzata soltanto nel primo '500 e importantissima per lo sviluppo della scienza algebrica moderna.

Le origini di tali ricerche da parte di Piero, si individuano nel matematico duecentesco Leonardo Pisano, detto il Fibonacci. In questo ramo della matematica le ricerche di Piero, anche se non coronate da pieno successo, conseguirono un buon risultato, tale da metterlo all'avanguardia dei matematici del suo tempo.

2. *Il contenuto geometrico dell'« Abaco »*

La parte geometrica dell'*Abaco,* in un certo senso, è la chiave per tutta la trattatistica di Piero. Sono gli esercizi geometrici a determinare gli effettivi collegamenti con il suo *Libellus* e la sua *Prospectiva pingendi.* Mentre il libro dell'abbaco, tipico del '400, è soprattutto un manuale pratico e tratta, per esempio, della misurazione di oggetti di carattere geometrico tratti dalla vita quotidiana, l'*Abaco* di Piero invece è concepito in una maniera tutta diversa. La sua ideazione è severa, chiara, precisa ed esatta, staccata nettamente dalla realtà contingente. Su un piano esclusivamente teorico viene trattata la misurazione di poligoni e poliedri di carattere geometrico ed astratto.

3. *La « sezione aurea » e i cinque corpi regolari nell'« Abaco » di Piero*

Nella trattazione di questi poligoni e poliedri due temi ci colpiscono: la « sezione aurea » e i corpi regolari. Anche qui le fonti delle ricerche di Piero sono da identificarsi nel Fibonacci e in Euclide. È importante riconoscere che Piero tratta la proporzione « avente il meçço et doi stremi » senza mai accennare al carattere particolare di essa. Questa proporzione della matematica antica era indispensabile alla costruzione euclidea del pentagono e conseguentemente a quella del

dodecaedro, e trovava il suo naturale sbocco nella dottrina dei cinque corpi regolari, così importante per i metodi costruttivi di Piero. Più tardi, nell'opera del Pacioli, la « sezione aurea » viene denominata « divina » e acquista quegli ipertoni mistico-speculativi tipici della sua applicazione alla teoria dell'arte. Tali elementi però mancano del tutto nel puro e rigoroso trattato matematico di Piero. Anche la trattazione dei corpi regolari procede su basi euclidee, e qui Piero getta le basi per il *Libellus de quinque corporibus regularibus,* un'opera ancora più prettamente euclidea.

4. *L'« Abaco » e la « Prospectiva pingendi »*

Anche i tre libri del trattato di Piero, *De prospectiva pingendi*, posano su fondamenti euclidei. Benché sia apparentemente un'opera di matematica pura, l'*Abaco* ha pure una sua attinenza alle arti figurative. È precisamente la misurazione delle superfici di corpi geometrici che forma la base della scienza prospettica. « *Prospectiva* » si chiama anche « *Commensuratio* » e, nel linguaggio pieriano, questa commensurazione « *diciamo essere essi profili et contorni proportionalmente posti nei luoghi loro* ». Il primo libro della *Prospectiva* tratta dei punti, delle linee e dei piani, e le definizioni di un abbaco vengono riformulate per l'artista. Qui si ricorda appunto la opera *Elementi di pittura* dell'Alberti, ma in Piero il discorso è condotto matematicamente fin nei minimi particolari. In confronto l'opera dell'Alberti sembra un lavoro da dilettante.

Il secondo libro della *Prospectiva pingendi* tratta dei corpi stereometrici e della loro costruzione. Nell'introduzione la relazione con l'*Abaco* si chiarisce: « *Corpo ha in sé tre dimensioni: longitudine, latitudine et altitudine; li termini suoi sono le superficie. I quali corpi sono de diverse forme, quale è corpo chubo, quale tetragono che non sono de equali lati, quale è tondo, quale laterato, quali piramide laterate, et quale di molti*

et diversi lati, sicommo ne le cose naturali et ancidentali se vede. De li quali in questo secondo [libro] intendo tractare de la loro degradationi, nelli termini posti da l'occhio socto angoli compresi, facendo de alcune superficie degradate nel primo lor base ».

In seguito il terzo libro tratta della realizzazione prospettica delle teste e delle strutture architettoniche. Qui il discorso si avvicina ancora di più alla pittura: « *Et perché la pictura non è se non dimostrationi de superficie et de corpi degradati o accresciuti nel termine, posti secondo che le cose vere vedute da l'occhio socto diversi angoli s'apresentano dicto termine...* ».

3.

Il « Libellus de quinque corporibus regularibus »: corpi regolari e irregolari

Il *Libellus* (Vat. Urb. lat. 632) fu steso fra il 1482 e il 1492, l'anno in cui morì Piero. La dedica al giovane duca Guidobaldo da Montefeltro si basa su una formula vitruviana (cfr. *De architectura,* Lib. III, prefazione) ed inoltre svela i motivi e gli scopi dell'autore nello scrivere il trattato, e in particolare il suo desiderio di collocarlo accanto all'altro trattato, il *De prospectiva pingendi,* che era stato dedicato al padre, ormai morto, del nuovo duca Guidobaldo, chiara testimonianza di una sostanziale continuità di struttura e tema fra i due trattati. Il successo più notevole di Piero, nel nuovo trattato, fu l'applicazione di princìpi aritmetici ai teoremi geometrici di Euclide, un risultato già preparato nel suo *Trattato d'abaco,* e poi proseguito con nuove riflessioni sul libro XV degli *Elementi* di Euclide. A questo punto vengono esposti i molteplici rapporti intercorsi fra il *Libellus* e l'*Abaco,* tanto sul piano del confronto dei testi, quanto a livello contenutistico.

1. *Corpi regolari: il « Libellus », parte I-III*

Più della metà degli esercizi del *Libellus* (I-III) si trovano già nella parte geometrica dell'*Abaco* (foll. 80r-120r).

Nel *Libellus,* I (dove si tratta di poligoni bidimensionali), 33 dei 55 esercizi si ripetono dall'*Abaco* (nn. 3, 4, 8, 11, 14-25, 28-30, 34-40, 42-47, 49). Di questi, 16 sono rielaborati (nn. 3, 14, 20, 24, 25, 28, 29, 35-41, 43, 44) e 17 riportati senza cambiamenti di rilievo (nn. 4, 8, 11, 15-19, 21-23, 30, 34, 45-47, 49). In questa prima parte del *Libellus* si manifestano inoltre nuovi riferimenti a Euclide negli esercizi 14, 34, 39 e 46.

Libellus, II, insegna la misurazione dei corpi regolari posti nella sfera. Dei 37 esercizi, 26 sono già nell'*Abaco*, 14 sono rielaborati (nn. 4, 6, 11, 12, 14-16, 18, 21, 26-28, 30, 35), e 12 sostanzialmente inalterati (nn. 5, 7-10, 13, 29, 31-34, 36).

Nel *Libellus,* III, iniziano le novità, specialmente nei primi esercizi (1-13), che riflettono una connessione profonda con il libro XV degli *Elementi* di Euclide. I problemi successivi, sulla sfera (14-29), sono quelli promessi da Piero nella sua introduzione al *Libellus*. Di questi 16 esercizi sulla sfera, 13 sono già nell'*Abaco* (nn. 15-17, 19-25, 27-29), 5 in una forma quasi identica (nn. 17, 23-25, 28) e 8 rielaborati (nn. 15, 16, 19-22, 27, 29), a volte con soluzioni alternative. Altri problemi dimostrano nuove utilizzazioni di Euclide (nn. 23, 25, 27, 28) e di Archimede (nn. 15, 16, 20, 23).

Bisogna riconoscere, però, che nel riproporre i problemi dell'*Abaco*, Piero ridefinisce le soluzioni, sia in termini di metodo matematico, sia in termini di esattezza numerica, con il risultato di un testo molto più chiaro e conciso. Si vedano, ad esempio, nel *Libellus*, gli esercizi I, 28 e II, 30, confrontandoli con i corrispondenti problemi nell'*Abaco*. Tuttavia la maggior precisione del *Libellus*, dal punto di vista stilistico, sembra largamente dovuta all'intervento del suo traduttore in latino.

Le fonti delle novità contenute nel *Libellus,* III, si identificano principalmente in nuovi studi fatti sul libro XV degli

Elementi di Euclide. Gli esercizi 1-13 trattano della misurazione di un corpo regolare posto in un altro, materia non incontrata nell'*Abaco,* e costituiscono quindi un'ulteriore tappa nello sviluppo matematico dell'artista.

Difatti all'inizio di questa « *tertia pars* », Piero effettua una operazione di quantificazione dalle proporzioni geometriche fisse esistenti fra i corpi regolari, l'uno iscritto nell'altro, derivandone numerosi esercizi matematici. Quasi tutti i problemi hanno una loro derivazione immediata da Euclide, corrispondenza specifica in una tavola a pagina 57.

2. *Corpi irregolari: il « Libellus de quinque corporibus », Parte IV*

Nel *Libellus*, IV, si passa alla considerazione di corpi irregolari. Gli esercizi per la maggior parte sono di natura prettamente matematica e di carattere assai astratto. Tuttavia qualche eccezione si collega a temi architettonici e, nell'insieme, la materia ha una sua implicita componente prospettivistica. Come in precedenza, si possono individuare esercizi ricavati dall'*Abaco* (nn. 2-9, 12-15, 18).

Il *Libellus,* IV, inizia con la complessa figura di un corpo di 72 facce: 24 triangoli e 48 poligoni di quattro lati. Le implicazioni architettoniche di tale figura sono sviluppate esplicitamente nella *Divina proportione* di Luca Pacioli, che la riferisce all'emisfero e allo « *antico templo pantheon* », nominandola « *forma assai accomodata* » agli architetti nel fare tribune e altre volte. Tali forme diventano argomenti per prospettive architettoniche nella *Prospectiva pingendi* di Piero, dove sono anche illustrate le raffigurazioni in prospettiva di colonne di 8 e 16 facce, e « *una volta in crociera* », che, nel *Libellus,* trovano corrispondenze in analoghi esercizi di misurazione (colonna e volta, « *per modum crucis* »). Ancora una volta l'intreccio « prospettiva-misurazione di corpi irregolari »

diventa più esplicito nell'opera del Pacioli, come si vede nella *Divina proportione.*

Forse in modo meno evidente le applicazioni alla pittura della misurazione di poligoni e poliedri, regolari e irregolari si avvertono nella *Prospectiva pingendi,* ma ci sono. E, nello stesso modo, due argomenti centrali — la « sezione aurea » e i corpi regolari — collegano l'*Abaco* al *Libellus.* La passione di Piero per la misurazione esatta e la raffigurazione accurata delle forme solide della natura, opera come una forma unificatrice in tutti e tre i trattati.

La conclusione del capitolo prende in esame alcuni riflessi di simili corpi regolari e irregolari in vari rami del disegno: in architettura, in scenografia e nella produzione di tarsie, una attività allora molto fiorente. Nelle tarsie abbiamo di fronte un mondo irreale di strumenti scientifici e musicali, di motivi sacri e secolari di natura morta, tutti messi accanto a prospettive urbane, e con aggiunti astratti corpi stereometrici, in una visione che sembra la materializzazione delle aspirazioni dei trattati pieriani.

4.

Applicazioni dell'arte del misurare

Le invenzioni geometriche dell'*Abaco* laurenziano e del *Libellus* vaticano di Piero furono rese accessibili a un vasto pubblico soltanto attraverso le opere di Fra Luca Pacioli. Nel 1494 il Pacioli pubblicò gli esercizi stereometrici dell'*Abaco* nella conclusione della sua *Summa arithmetica*, senza però accennare al vero autore di essi. Più tardi, nel 1509, in modo del tutto simile, incluse il *Libellus* di Piero nella sua *Divina proportione*. Così i due trattati di Piero sui corpi regolari poterono più facilmente diventare la base per ulteriori sviluppi nella trattatistica artistica, ciò che poi è avvenuto non soltanto in Italia ma anche fuori. Questo fenomeno viene esaminato principalmente per quanto concerne gli studi e le ricerche sulla prospettiva.

I legami fra i corpi regolari e la prospettiva, non del tutto evidenti, si avvertono già nella *Prospectiva pingendi* di Piero, dove è lasciato intendere che lo studio di poligoni e poliedri geometrici costituisce il primo passo nell'apprendimento della prospettiva. Più tardi nelle pagine del Vasari leggiamo di un disegno prospettico molto elaborato di Piero, « *un vaso in modo tirato a quadri e facce, che si vede dinanzi, di dietro e dagli lati, il fondo e la bocca: il che è certo cosa stupenda, avendo in quello sottilmente tirato ogni minuzia, e fatto scortare il girare di tutti quei circoli con molta grazia* ». Questo

magnifico disegno esemplifica il profondo sapere di Piero, « *maestro raro* », che meglio di ogni altro intese « *tutti i migliori giri tirati nei corpi regolari* ».

Nella *Summa arithmetica* il Pacioli elenca vari artisti con i quali aveva discusso la prospettiva (Gentile e Giovanni Bellini, Andrea Mantegna, Melozzo da Forlì, Signorelli, Botticelli ed altri), « *quali sempre con libella e circino lor opere proportionando a perfection mirabile conducano* ». D'altronde egli consiglia loro la *Prospectiva pingendi,* un « *dignissimo compendio e per noi ben apreso* ». Insieme alla versione italiana del *Libellus* sui corpi regolari di Piero, nella *Divina proportione,* il Pacioli presenta cinquantanove grandi silografie di corpi geometrici, regolari e irregolari, disegnati in prospettiva.

La rapida diffusione delle idee e delle invenzioni di Piero è stata assicurata, in parte, dal particolare pubblico a cui il Pacioli indirizzava i suoi libri. Nella *Summa* egli indicò come destinatari: architetti, prospettivisti e intarsiatori. Nella *Divina proportione,* « *ciascun studioso di Philosophia, Prospectiva, Pictura, Sculptura, Architectura, Musica e altre Mathematice* ». Di questa opera di divulgazione si trova l'eco nel ritratto del Pacioli dipinto nel 1495, adesso nel Museo di Capodimonte a Napoli. Il quadro raffigura il dotto frate occupato in esercitazioni euclidee sui corpi regolari, che in un certo senso ripropongono gli insegnamenti di Piero.

Il Pacioli si sente come un maestro degli artisti e la sua opera non era certa inefficace. Difatti ci conduce, per prendere in esame soltanto tre esempi, ai trattati del grande Dürer e di un orefice suo conterraneo, il norimberghese Wenzel Jamnitzer, e, di nuovo in Italia, ai trattati di Daniele Barbaro.

Come tutto il lavoro teorico del Dürer, l'ultimo libro della sua *Unterweisung der Messung* si fonda su premesse italiane. Dedicata agli studi di prospettiva e dei poliedri, questa parte della *Unterweisung,* invece di occuparsi della misurazione stereometrica secondo il severo metodo pieriano, presenta modelli grafici dai quali si possono costruire in carta i cinque corpi

regolari, nove altri corpi irregolari e la sfera. Poi il Dürer illustra la raffigurazione prospettica di un corpo regolare, il cubo, e il metodo per metterlo in luce e ombra. Tale modello si presta ad una generalizzazione molto ampia, poiché il Dürer scrive che, avendo la pianta, ogni cosa si potrebbe disegnare nella stessa maniera. In questo modo le applicazioni pratiche dei corpi geometrici diventano vastissime, almeno sul piano teorico. Infatti il suo libro è pervaso dal costante riferimento alla pratica dell'artista e nella *Proportionslehre* il Dürer tratta perfino anatomie in termini geometrici.

L'opera del Dürer ispirò una lunga serie di piccoli trattati, i cosidetti *Kunstbüchlein* and *Perspectivbüchlein* tedeschi. Fra questi, nel 1568, apparve il libro dello Jamnitzer con un caratteristico titolo interminabile, la *Perspectiva corporum regularium, das ist ein fleyssige Fürweisung wie die Fünff Regulierten Cörper, darvon Plato in Timaeo unnd Euclides inn sein Elementis schreibt etc., durch einen sonderlichen newen behenden unnd gerechten Weg.....gar künstlich inn die Perspectiva gebracht....* Da una parte questo libro, pieno di una gran varietà di forme geometriche in magnifiche tavole di rame, rappresenta il culmine delle ricerche di Piero, del Pacioli e del Dürer, in quanto illustra tutti i cinque corpi regolari e in più centocinquanta varianti di essi. Dall'altra l'opera dello Jamnitzer è una semplificazione della tradizione dei grandi maestri. Nel testo manca ogni indicazione utile per la costruzione dei corpi geometrici, e non c'è niente o quasi delle teorie matematiche e proporzionali fondamentali ad essi relative. Qui abbiamo sostanzialmente esercizi accademici di stereometria utili nello studio della prospettiva. Così il libro dello Jamnitzer diventa un vero libro di modelli. Questo fatto, però, non cancellava completamente la sua origine. E fu Egnazio Danti, biografo del Vignola e noto matematico, che nel 1583 scrisse di « *Wenceslao Giannizzero Norinbergese, il quale hà messi in Prospettiva li corpi regolari, et altri composti, si come fece Pietro dal Borgo...* ».

Dopo la loro prima pubblicazione in volgare, i due trattati del Dürer, la *Unterweisung* e la *Proportionslehre,* uscirono in edizione latina, e furono resi così più accessibili alla cultura italiana del Cinquecento. In quel periodo la dottrina della prospettiva fu coltivata, in senso pratico, attraverso l'opera del dotto commentatore di Vitruvio e patriarca di Aquileia, Daniele Barbaro, la *Pratica della perspettiva* (Venezia 1569), opera divenuta celeberrima in Italia. Il Barbaro assorbì moltissimi concetti da Piero, ma la sua dimestichezza con il pensiero pieriano sembra derivare in parte dagli scritti del Pacioli e del Dürer. La sua *Pratica della perspettiva* parte da istruzioni molto particolareggiate per i poligoni e i poliedri, per giungere a prospettive architettoniche, un interesse scenico del tutto naturale per un teorico dell'architettura.

Meno nota della sua *Pratica della perspettiva* è la quasi omonima *Pratica della prospettiva,* sempre del Barbaro, rimasta in manoscritto (circa 300 pagine), adesso nella Biblioteca Marciana a Venezia. La *Prospettiva* non è altro che un trattato vero e proprio sui corpi regolari, e come tale la più esauriente trattazione mai scritta sull'argomento. Una breve introduzione ricapitola i princìpi basilari della prospettiva, ma il resto del lungo trattato è dedicato al disegno prospettico di corpi geometrici. Qui, come nel libro stampato, il Barbaro collega il tema con la raffigurazione di motivi architettonici e infine con la scenografia:

> Hor perche gli e necessario volendo per arti di prospettiva designari sopra uno quadro qualche architettura, over altra cosa che star debbia, come essa perspettiva richiede, intendo dir il suo perfetto, non dico perfetto quanto a tutta la forma: perche presupono che uno dipintor, oltre la perspettiva debba perfettamente intendere quali siano li buone forme: si di gli edificij, come da tutti li altre cose: volendo nel arte sua procedere secondo che ella richiede: ma solamente intendo per il suo perfetto haver cognitione che forma, che li habbia, cio è triangolare, quadrata, pentagona, over di altra sorte: et come tutti li anguli di essa architettura, over altra cosa essendo materialmente costruita.

E' presentata un'immensa varietà di poliedri e numerosi esempi di un corpo iscritto in un altro, per giungere a corpi elevati e stellati e mazzocchi complicatissimi. Un'altra parte del trattato si occupa della sfera, e anche questa appartiene alla pratica dell'artista: « ... et se un pittore volesse far in pittura un corpo spherico, over balla, tirra col compasso un circolo, et quello date le sue ombre diligentemente havrà fatto lo detto corpo spherico ». La massima parte del trattato è dedicata al disegno prospettico dei corpi geometrici, un argomento pieno di difficoltà, che comunque per il Barbaro merita tutta la fatica che richiede. Nel suo insieme l'opera rivela il fascino dei corpi regolari non meno dell'opera del Pacioli, ma nel Barbaro mancano le speculazioni filosofiche o metafisiche che si trovano nel Pacioli. I corpi geometrici vengono presentati invece come il vero fondamento per la corretta rappresentazione di spazio e volume nelle arti figurative.

Appendice 1.

Luca Pacioli e il «Trattato d'abaco» di Piero

Nella *vita* di Piero, il Vasari scrisse: « *venuto Piero in vecchiezza ed a morte, dopo avere scritto molti libri, maestro Luca [Pacioli] detto, usurpandoli per se stesso, gli fece stampare come suoi, essendogli pervenuti quelli alle mani dopo la morte del maestro* ». L'accusa del Vasari al Pacioli di aver plagiato i lavori di Piero, suscitò numerosi echi, a volte non senza accenti polemici, principalmente nei commenti alle varie edizioni dell'opera vasariana. Per risolvere la questione, lungamente discussa, mancava assolutamente ogni dato concreto. Soltanto nel 1880 il tedesco Max Jordan, avendo rintracciato il manoscritto del *Libellus de quinque corporibus regularibus* di Piero nella Biblioteca Vaticana, riconobbe in esso la versione latina di un trattato italiano, intitolato *Libellus in tres partiales tractatus divisus quinque corporum regularium et dependentium,* pubblicato in appendice nella *Divina proportione* (1509) dal Pacioli, avendo così la conferma che il Pacioli si era appropriato del lavoro di Piero. Ma non si deve dimenticare che il concetto del plagio, definito in senso moderno, è piuttosto tardo, e che nel Rinascimento tale concetto era ancora piuttosto labile.

Alle dimostrazioni dello Jordan si può aggiungere un altro caso. L'ultimo capitolo della *Summa de arithmetica, geometria, proportioni et proportionalità* del Pacioli (Venezia

1494) si intitola: *Particularis tractatus circa corpora regularia et ordinaria.* È un trattato stampato su una decina di grandi pagine che concerne la misurazione dei corpi regolari e delle loro varianti irregolari. Questo trattato, con una dedica a Guidobaldo da Montefeltro tutta sua, non rappresenta altro che una serie degli esercizi dell'*Abaco* di Piero (foll. 105r-120r), e cioè gli esercizi stereometrici discussi sopra nel capitolo II. 27 problemi, più della metà, sono riportati quasi alla lettera (nn. 3-5, 11, 14, 16, 19, 20, 25, 27-30, 32, 33, 35, 37-44, 46, 54, 55); 10 diventano un po' più concisi (nn. 6, 9, 10, 13, 15, 17, 21, 26, 34, 48); 6 abbreviati in modo notevole (2, 7, 8, 22, 52, 56); 3 allungati per maggiore chiarezza (12, 45, 47); 4 parzialmente rielaborati (18, 15, 53, 53 bis); e soltanto cinque rivisti interamente (23, 24, 36, 49, 50). Tutte queste corrispondenze sono elencate nella Tavola I.

Più tardi nella *Divina proportione* il Pacioli riferisce la sua intenzione di non trattare della misurazione dei corpi regolari, « *per haverne già composto particular tractato* », indubbio accenno al *Particularis tractatus* della *Summa,* e così rivendicando a sé la paternità dell'opera. Forse più importante, però, è riconoscere nella materia del *Particularis tractatus* una fase preliminare del *Libellus de quinque corporibus regularibus* di Piero.

Appendice 2.

I manoscritti di Piero e gli adattamenti del Pacioli

Da un attento esame dei quattro testi relativi alla questione Piero-Pacioli varie conclusioni risultano verosimili. Sembra evidente che Piero componesse il testo originale del suo *Libellus de quinque corporibus regularibus* in italiano, per lasciarlo tradurre in latino da un umanista amico, molto probabilmente da identificarsi nella persona di Matteo da Borgo. A sua volta la versione italiana pubblicata dal Pacioli nel 1509 (il *Libellus in tres partiales* della *Divina porportione*) rappresenta una rivolgarizzazione del *Libellus* vaticano di Piero, e non la versione italiana primitiva.

Come è stato dimostrato in precedenza una gran parte del *Libellus* vaticano di Piero era già presente nel suo *Abaco*. Ciò nonostante, ci sono divergenze di sostanziale portata fra le due opere, e quella più tarda sviluppa certamente una matematica più evoluta. È impossibile che il Pacioli non si sia accorto di un tale sviluppo. Tuttavia nei suoi prestiti clandestini il Pacioli non si preoccupò delle disparità fra i due testi. Prima pubblicò il trattato più maturo, e poi quello meno sviluppato, invertendo così l'ordine logico dell'opera pieriana. Certo il Pacioli dovette tradurre o far tradurre uno dei manoscritti di Piero per divulgarne il contenuto a un più largo pubblico, ma di pari importanza è, forse, il desiderio di una maggiore diffusione della propria opera letteraria per mezzo di una calcolata politica editoriale.

I.

Artists and Mathematicians: Arithmetical, Geometrical, and Perspectival Studies

Even though it is universally agreed that geometry is a central element of Piero's painting, his two mathematical treatises, the *Trattato d'abaco* and the *Libellus de quinque corporibus regularibus*, are known only superficially, if at all, to art historians, who pay them little more than the lip-service of a passing mention [1]. There exists no study of their contents, sources, and significance that places them within the context of Piero's own art and mathematics, as well as in the context of the art and mathematics of Piero's time [2]. This is perhaps understandable, for, unlike the precepts contained in Piero's *De prospectiva pingendi,* the theorems in

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[1] For Piero and Piero studies, see the comprehensive monograph by EUGENIO BATTISTI, *Piero della Francesca,* 2 vols. (Milan 1971).

[2] Three treatises by Piero have come down to us, the *Trattato d'abaco* (Florence, Biblioteca Medicea Laurenziana, MS. Ashb. 280/359-291), the *Libellus de quinque corporibus regularibus* (Rome, Biblioteca Vaticana, MS. Vat.Urb.lat. 632), and the *De prospectiva pingendi* (Parma, Biblioteca Palatina, MS. 1576, in Italian; Milan, Biblioteca Ambrosiana, Cod. Ambr. C. 307, in Latin). All have been published except the Latin version of the *Prospectiva pingendi*: *Trattato d'abaco,* ed. G. Arrighi (Pisa 1970); *L'opera « De corporibus regularibus » di Pietro Franceschi detto della Francesca usurpata da Fra Luca Pacioli,* ed. G. Mancini (Rome 1916); *De prospectiva pingendi,* ed. C. Winterberg (Strassburg 1899); *De prospectiva pingendi,* ed. G. NICCO FASOLA, 2 vols. (Florence 1942).

these two works do not seem to apply directly to painting. Nonetheless, this lacuna impedes a comprehensive evaluation of Piero's artistic achievement, for the *Trattato d'abaco* and the *Libellus* relate to perspective study and to the concerns of the Renaissance painter. They initiate a specific theme which emerges more fully developed in later manuals for artists. Seen in this light, Piero's two mathematical treatises form the fountain head of one branch of the literature of art, the territory first charted in Schlosser's *Kunstliteratur.*

* * *

Piero's two-sided interest in mathematics and art represented nothing out of the ordinary. The Renaissance witnessed a flourishing union between the fine arts and the mathematical sciences of arithmetic and geometry. Its first fruits are visible in Trecento painting, and Piero's *Abaco* and *Libellus,* as well as his *Prospectiva pingendi,* are manifestations of the same union. Indeed by the mid-fifteenth century the collaboration of artists and mathematicians in artistic undertakings was an everyday occurrence, and, by this time, treatises by artists evidence their independent investigations of arithmetical and geometrical principles [3]. Subsequently, with the foundation of academies in the sixteenth century, this traditionally informal association was institutionalized, and mathematics became part of the curriculum artists studied [4].

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[3] For the technical and mathematical writings of ALBERTI, GHIBERTI, FILARETE, FRANCESCO DI GIORGIO, PIERO DELLA FRANCESCA, LEONARDO AND ALBRECHT DÜRER, see L. OLSCHKI, *Geschichte der neusprachlichen wissenschaftlichen Literatur* (1919; rpt. Vaduz 1965), vol. I. Also H. BROCKHAUS, *De Sculptura von Pomponius Gauricus* (Leipzig 1886), pp. 36-58; M. BOSKOVITS, « Quello ch'e dipintori oggi dicono prospettiva: Contributions to Fifteenth-Century Italian Art Theory, » *Acta historiae artium Academiae Scientiarum Hungaricae,* VIII (1962), 241-60; IX (1963), 139-62; P. TIGLER, *Die Architekturtheorie des Filarete* (Berlin 1963), esp. pp. 33-39, 58-65, 68, 175-77.

[4] See OLSCHKI, *Geschichte,* II, 171-99; [Girolamo Ticciati] « Sto-

Still nearly all mathematical study by Renaissance artists deserves to be viewed from the vantage point of a situation in which artists and mathematicians studied together and learned together. Although the mathematicians often undeniably furnished the lead, this background is as revealing for artists who developed their mathematical skills independently as for those who were trained and encouraged to formulate mathematical inventions by mathematician teachers.

Ghiberti, whose *Commentari* document his own scientific interests, draws attention to Trecento artists whose paintings reflect theoretical concerns. He describes one follower of Giotto, a certain Stefano, « *egregissimo dottore* », and says that his paintings were executed with « *grandissima dottrina* ». Among the learned Sienese masters, Ghiberti counts Ambrogio Lorenzetti, « *molto perito nella teorica di detta arte* », the art of *disegno* [5]. And it is in Lorenzetti's *Annuciation* of 1344 that the first known instance of the use of a single vanishing point occurs. This invention was predicated on theoretical studies, and it would certainly have earned Ambrogio a reputation for being learned [6]. Later, Vasari reports that as a youth Ambrogio dedicated himself to the study of literature and that in his later years he sought out the company of educated men [7]. Owing to Ambrogio's optical accomplishments, his educated friends have primarily been identified as mathematicians and students of optics [8].

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ria della Accademia del Disegno, » in P. FANFANI, *Spigolatura michelangiolesca* (Pistoia 1876), pp. 269-72.

[5] *I commentari,* ed. O. Morisani (Naples 1947), pp. 34, 38.

[6] G. J. KERN, « Die Anfänge der zentralperspektivischen Konstruktion in der italienischen Malerei des 14. Jahrhunderts, » *Mitteilungen des Kunsthistorischen Institutes in Florenz,* II (1912), 58-60; E. PANOFSKY, « Die Perspektive als symbolische Form, » *Vorträge der Bibliothek Warburg,* 1924-25 (Leipzig 1927), p. 279.

[7] VASARI-MILANESI, I, 524.

[8] KERN, « Anfänge, » p. 63.

From Antonio Manetti's biography we learn that Brunelleschi's friendship with the mathematician Paolo dal Pozzo Toscanelli extended over some forty years [9]. And, elsewhere it is recorded that Toscanelli collaborated with Leon Battista Alberti in joint astronomical observations [10]. Most likely, Brunelleschi, too, learned from Toscanelli and had him to thank for many of the principles that lay at the foundation of his engineering and perspectival feats [11].

Another representative figure, Luca Pacioli, was born in Borgo Sansepolcro, the home town of Piero, and he became one of the most noted fifteenth-century mathematicians. In what must have been a familiar occupation for a mathematician of his day, Luca dal Borgo, as Pacioli was often called, taught arithmetic and geometry to the architects and stone ma-

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[9] A. MANETTI, *Vita di Filippo Brunelleschi,* ed. D. De Robertis (Milan 1976), p. 70.

[10] See M. CURTZE, « Der Briefwechsel Regiomontan's mit Giovanni Bianchini, Jacob von Speier und Christian Roder, » *Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen,* XII (1902), 263-64, for a letter from Regiomontanus to Giovanni Bianchini: « *Item tempore nostro oporteret declinationem Solis maximam esse g 24 m 2. Nos autem (preceptor meus et ego) instrumentis reperimus eam g 23 m 28 fere. M. Paulum Florentinum et D. Baptistam de Albertis sepe audivi dicentes, se diligenter observasse et non reperisse maiorem g 23 m 30, que res etiam tabulas nostras, videlicet tabulam declinationis et ceteras, que supra eum fundantur, innovare persuadet.* »

[11] For Brunelleschi's relationship to Toscanelli see most recently S. Y. EDGERTON, JR., *The Renaissance Rediscovery of Linear Perspective* (New York 1975), pp. 61-63, 121-23. Also: VASARI-MILANESI, II, 333; OLSCHKI, *Geschichte,* I, 45. For Toscanelli's relationship to contemporary artists, see further: H. BROCKHAUS, *Über die Schrift des Pomponius Gauricus « De Sculptura, » Florenz 1504* (Leipzig 1885), pp. 36-38, 37, n. 1; G. UZIELLI, *Intorno ad un passo di Giorgio Vasari relativo a Paolo dal Pozzo Toscanelli, quale maestro di Filippo Brunelleschi* (Rome 1894); idem, *Ricerche intorno a Leonardo da Vinci,* 2nd ed. (Turin 1896), I, passim; idem, *La vita e i tempi di Paolo dal Pozzo Toscanelli* (Rome 1894). For Brunelleschi's education, P. SANPAOLESI, « Ipotesi sulle conoscenze matematiche, statiche e meccaniche del Brunelleschi, » *Belle arti* (1951), pp. 25-54.

sons of his native town. And, stimulated by their demand for more precise mathematical rules and methods to employ in architectural undertakings, he composed for them a treatise based largely on his independent study of Vitruvius [12]. Beyond this, Pacioli formed a close relationship with Leonardo da Vinci when both were at the court of Lodovico il Moro in Milan. Pacioli's treatises and Leonardo's notebooks testify to their association. Pacioli knew Leonardo's plans for the Sforza monument in detail, for he relates the weight and measurement of the colossal bronze horse [13], and he acknowledges that Leonardo was responsible for the designs of the stereometric bodies appended to his *Divina proportione* [14]. In turn, Leonardo acquired a copy of Pacioli's *Summa de arithmetica, geometria, proportioni et proportionalità* upon its appearance in 1494, and he made a note to himself in the Codice Atlantico to learn the multiplication of square roots from Pacioli: « *Impara la multiplicazione delle radici da maestro Luca* »[15].

Tuscany, the traditional home of *disegno,* was not the only province which witnessed the marriage of art and mathematics. In the Veneto, Jacopo Bellini's perspective experiments were certainly encouraged by the scientist Giovanni Fontana, who dedicated his now lost treatise on perspective to the Ve-

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[12] L. Pacioli, *Divina proportione* (Vènice 1509), fol. 23r: « *Ali suoi caris... discipuli e alievi... Essendo da voi più volte pregato che oltre la prathica de Arithmetica e Geometria datovi insiemi ancora con quelle dar vi volesse alcuna norma e modo a poter conseguire el vostro disiato effecto delarchitectura ...* »

[13] *Divina proportione,* fol. 1r: « *Commo ladmiranda e stupenda equestre statua. La cui alteçça dela cervice a piana terra sonno bracia 12 cioe 37 4/5 tanti dela qui presente linea .ab. e tutta la sua ennea massa alire circa 200000 ...* »

[14] Pacioli, *Divina proportione,* Dedication, and fols. 22r, 28v, 30v. A further notice in Pacioli's « De viribus quantitatis, » Bologna, Bibl. Univ., MS. 250, fol. 237r.

[15] E. Solmi, *Le fonti dei manoscritti di Leonardo da Vinci* (Turin 1908), pp. 220-21.

netian painter. Earlier Fontana had studied under Biagio Pelacani, an optics theorist, who lectured at the University of Padua between 1377 and 1411, and Fontana himself had probably begun writing treatises as early as 1420 [16]. Still at Venice, the almost Tuscan precision of Carpaccio's perspective constructions have been remarked more than once, and a recent writer has put the matter pointedly, by asking how the Venetian painter learned the synthetic spatial perspective of Piero della Francesca [17]. A notice contained in a manuscript by Daniele Barbaro, the sixteenth-century *trattatista,* establishes that the principles of perspective were taught to Vittore Carpaccio and Giovanni Bellini by Hieronymo Malatini, whom Piero's follower, Luca Pacioli, describes as a perspective master [18]. And Barbaro tells that he himself learned the principles of perspective from the mathematician Giovanni Zamberti [19], who, it happens, was the brother of Bartolomeo Zamberti, the first editor and translator of Euclid from Greek into Latin [20]. Thus the circle turns back upon itself, and it is telling

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[16] L. THORNDIKE, *A History of Magic and Experimental Science,* IV (New York 1934), 155-57. Pelacani's *Quaestiones perspectivae* (1390) was, it seems, owned by Paolo dal Pozzo Toscanelli. See R. KLEIN, « Pomponius Gauricus on Perspective, » *Art Bulletin,* XLIII (1961), 211.

[17] *Vittore Carpaccio: catalogo della mostra,* ed. P. Zampetti (Venice 1963), p. XXXIII.

[18] D. BARBARO, « La pratica della prospettiva, » Venice, Bibl. Marciana, MS. It. IV, 39 (=5446), fol. 2r; L. PACIOLI, *Summa de arithmetica, geometria, proportioni et proportionalità* (Venice 1494), Dedication. Malatini's name recurs in connection with an engineering project, dated 1489, for the river Brenta in the Codex Zichy, Budapest, Erwin Szabo Library, fol. 199. SEE M. AZZI VISENTINI, « Riflessioni su un inedito trattato di architettura: il codice Zichy della Biblioteca Comunale di Budapest, » *Arte veneta,* XXIX (1975), 139-45, fig. 6.

[19] D. BARBARO, *La pratica della perspettiva* (Venice 1569), p. 2: « *Il caso mi portò dinanzi un Giovanni Zamberto cittadino Vinitiano, il quale io ho usato per guida nella pratica della Perspettiva, & ho preso da quello molte cose, che mi sono state utili, & di piacere.* »

[20] See Zamberti's dedication of Euclid's treatise on optics in *Euclidis*

that in all three cases perspective procedures were transmitted not by an artist but by a specialist mathematician. In this respect, it is worth remembering the still nameless mathematician who revealed the « secrets » of perspective to Albrecht Dürer, for these theoretical principles were so important that they induced Dürer to undertake the journey from Venice to Bologna to learn them [21].

Where artists and their own mathematical studies and writings are concerned, we may discover the proper context of Piero's treatises by examining a sample of the mathematical-scientific interests manifested by his fellow artists. Here we may begin by turning again to Dürer. From his own notes, we learn that Dürer purchased Zamberti's edition of Euclid's *Elements* in Venice in 1507, and his continuing involvement with this work is not difficult to trace throughout his treatises [22]. The importance he attached to geometry is epitomized in the dedication of his *Unterweisung der Messung,* where he writes:

> Die weyl aber die [Kunst der Messung] der recht grundt ist aller mallerey, hab ich mir fürgenomen allen künstbegyrigen jungen, eyn anfang züstellen und ursach zügeben damit sie sich der messunge zirckels und richtscheyt underwinden unnd darauss die rechten warheyt erkennen unnd vor augen sehen mögen, damit

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megarensis philosophi platonici ... Elementorum libros XIII ..., ed. B. Zamberti (Venice 1505): « *Bartholomaeus Zambertus venetus Ioanni Zamberto Veneto frati humanissimo salutem perpetuam.* »

[21] E. PANOFSKY, *Albrecht Dürer,* 3rd ed. (Princeton 1948), I, 251; *Albrecht Dürer 1471-1971: Ausstellung des Germanischen Nationalmuseums, Nürnberg* (Munich 1971), pp. 341-42.

[22] *Dürer: Schriftlicher Nachlass*, ed. H. Rupprich, I (Berlin 1956), 221. The Zamberti edition of Euclid's *Elements*, owned by Dürer, contained the following annotation: « *Daz puch hab ich zw Venedich vm ein Dugatn kawft im 1507 jor. Albrecht Dürer.* » SEE DÜRER: *Nachlass,* II (1966) and III (1969), index, s.v. « Euklid, » for his references to the *Elements.*

> sie nit alleyn zü künsten begirig werden, sonder auch zu eynem rechten und grösseren verstant komen mögen [23].

Returning to Florence, at the very beginning of the Renaissance, one encounters the third book of Ghiberti's *Commentari,* a work which although it treats optics and the proportions of the human figure, is often considered little more than a confusing jumble of quotations. A recent penetrating analysis of Ghiberti's third *Commentario* demonstrates, however, that Ghiberti's selection of passages from medieval aptics treatises is not random, but purposeful and systematic, and intended to give a scientific basis to the study and use of light in the arts of sculpture and glass-painting [24].

The most prominent *trattatista* following Ghiberti is Leon Battista Alberti, whose works are noted for their eminently scientific spirit. Alberti's *De' ludi mathematici,* a work of plane geometry, presents fairly common exercises in such problems as the correct measurement of towers, rivers, and fields [25]. His *Elementi di pittura* is a plane and solid geometry treatise, but its mathematical definitions have been reworked for artists. It begins in a Euclidean fashion by defining the point: « *El puncto dicono essere quello che nulla si possa dividere in parte alcuna* ». For artists this is then redefined: « *Puncto nominamo noi in pictura quella picola jnscriptione, quale nulla puote essere minore* ». And an outline is defined thus: « *Lembo nominamo tuta quella circumducta descriptione facta da linee sotilissime per quale sia l'una area divisa da l'altra* ». The artist, significantly, is not instructed to draw after

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[23] *Underweysung der messung mit dem zirckel und richtscheyt* (1525; facsimile rpt. Dietikon-Zürich 1966), verso of title page.

[24] J. GAGE, « Ghiberti's *Third Commentary* and its Background, » *Apollo,* XCV (1972), 364-69, esp. 368.

[25] « De' ludi matematici, » in *Opere volgari di Leon Batt. Alberti,* ed. A. Bonucci, IV (Florence 1847), 405-40.

nature, but rather he is presented with abstract regular and irregular polygons to the end of representing three dimensional forms in perspective. The reader is urged to pursue these rudiments of painting diligently until he has grasped them. They are, Alberti writes, striking the double chord of the *utile* and the *dulce,* no less useful than amusing [26].

Even more direct observations concerning the nature of the relationship between geometry and art are found in Alberti's *Della pittura.*

> Piacemi il pittore sia dotto in quanto e possa in tutte l'arti liberali ma imprima desidero sappi giometria ... I nostri dirozzamenti, da i quali si exprime tutta la perfetta absoluta arte di dipigniere, saranno intesi facile dal geometra, ma a chi sia ignioranti in geometria né intenderà quelle né alcun'altra ragione di dipigniere: pertanto affermo sia necessario al pittore inprendere geometria [27].

Thus the painter must have a scientific understanding of what he sees and a knowledge of geometry to reproduce it correctly. Thereby naturalism is given a scientific basis, and, too, a path is indicated whereby modern artists may surpass those of antiquity.

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While Piero's interest in mathematics was traditional for an artist, his treatises, from the point of view of their content and their inspiration, are also a function of the rebirth of clas-

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[26] « Elementi di pittura, » in *Opera inedita et pauca separatim impressa,* ed. G. Mancini (Florence 1890), pp. 47-65. See also A. GAMBUTI, «Nuove ricerche sugli Elementa picturae, » *Studi e documenti di architettura* (1972), no. 1, 133-72; G. ARRIGHI, « Leon Battista Alberti e le scienze esatte, » *Convegno internazionale indetto nel V Centenario di Leon Battista Alberti, Roma — Mantova — Firenze, 25-29 aprile 1972* (Rome 1974), pp. 155-212.

[27] *Della pittura,* ed. L. Mallè (Florence 1950), pp. 103-04.

sical studies [28]. Piero was undoubtedly familiar with Euclid, Archimedes, and Ptolemy, for he mentions them and quotes from their works [29]. Moreover, his familiarity with Vitruvius can, not surprisingly, be demonstrated in the dedication to the *Libellus* [30]. Vitruvius' *De architectura,* together with Pliny's *Naturalis historiae,* constituted the principal literary sources from which the Renaissance sought to rediscover the underlying principles of ancient art. Both prescribe the knowledge of mathematics as a fundamental prerequisite to the practice of art. Among the many accomplishments of Vitruvius' ideal educated architect, one was mathematics. And without the aid of arithmetic and geometry, according to Pliny, art will never attain perfection [31]. In this context Piero's *Abaco* and his *Libellus* must be seen as aids for the Renaissance painter who attempted to follow a programme *all'antica* in the pursuit of artistic excellence.

It is in this context of the close associations between artists and mathematicians, a union sanctioned, even prescribed by classical antiquity, that the emergence of Piero's treatises belongs. Furthermore, for Piero the art of painting could almost be subsumed under the mathematical sciences. This view was not without opponents, for Piero himself writes that many painters still fail to understand the true significance of lines and angles, and they consequently disapprove of perspective [32].

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[28] BATTISTI, *Piero della Francesca,* I, 457, n. 1.

[29] In the *Trattato d'abaco* Euclid is referred to on fols. 87v; 90r-v; 103r; 104r-v; 106r; 110r-v; 112r-v; Ptolemy on fols. 98r and 110v; and Archimedes on fol. 106v. I follow here, and in the following pages, Arrighi's accurate transcription and his folio numeration.

[30] See below Chapter III, pp. 44-45.

[31] VITRUVIUS, *De architectura* (I, i, 4) and PLINY, *Naturales historiae* (XXXV, xxxvi, 76). In this connection see also L. LANZI, *Storia pittorica dell'Italia,* I (Bassano 1795), 49-50: « *Pietro della Francesca ... par che fosse il primo a richiamar l'uso dei Greci, che la geometria fecero servire alla pittura.* »

[32] PIERO DELLA FRANCESCA, *De prospectiva pingendi,* ed. G. Nicco

In his *De prospectiva pingendi* Piero defines painting as a part of perspective, that is to say as a branch of geometry.

> Et perché la pictura non è se non dimostrationi de superficie et de corpi degradati o acresciuti nel termine, posti secondo che le cose vere vedute da l'occhio socto diversi angoli s'apresentano nel dicto termine, et però che d'onni quantità una parte è sempre a l'ochio più propinqua che l'altra, et la più propinqua s'apresenta sempre socto magiore angolo che la più remota nei termini assegnati, et non posendo giudicare da sé lo intellecto la loro mesura, cioè quanto sia la più propinqua et quanto sia la più remota, però dico essere necessaria la prospectiva, la quale discerne tucte le quantità proportionalmente commo vera scientia, dimostrando il degradare et acrescere de onni quantità per forza de linee [33].

Painting presupposes the ability to measure solid objects and to represent them scientifically, thus requiring a solidly founded knowledge of the principles of arithmetic and geometry. Both of Piero's minor treatises, the *Trattato d'abaco* and the *Libellus de quinque corporibus regularibus,* testify to Piero's remarkable accomplishments in this area. Pure mathematics they are, but they are more than that, for as we will see they are immediately connected with the artistic concerns of the day.

* * *

Piero's *Trattato d'abaco* belongs to a genre of mathematical texts rarely considered by students of the arts, yet the *abbaco* perhaps represented the first important means whereby

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Fasola, p. 128: « *Molti dipintori biasimano la prospectiva, perché non intendano la forza de le linee et degli angoli, che da essa se producano.* »

[33] *Prospectiva pingendi,* pp. 128-29. See also the discussion of painting as a practical branch of mathematics in M. Savonarola's « Libellus de magnificis ornamentis regiae civitatis Paduae, » in L. A. Muratori, *Rerum Italicarum scriptores* XXIV (Milan 1738), col. 1169.

artists became acquainted with mathematical learning. Its initial purpose was to teach commercial mathematics to future merchants, but in time, with the addition of a geometry section (which saw successive refinements), the *abbaco* books became more ambitious in scope and suitable for a wider public [34].

The *abbaco* texts borrow extensively from one another, but all are ultimately based on the *Liber abbaci* (1202) and the *Practica geometriae* (1220) of Leonardo Pisano, who is commonly called Fibonacci [35]. Singlehandedly, Fibonacci revolutionized medieval mathematics by introducing Arabic numerals and methods of computation, as he said, « to the Latin races in order that they no longer be ignorant in these things » [36]. The Arabic inventions that he introduced were condensed, translated into Italian, popularized in the *trattati dell'abbaco,* and taught in the *scuole dell'abbaco* of the fourteenth and fifteenth centuries [37].

An education in a *scuola dell'abbaco* entailed learning the rudiments of arithmetic and geometry. It apparently preceded apprenticeship in a trade, for the average age of the students,

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[34] See R. Goldthwaite, « Schools and Teachers of Commercial Arithmetic in Renaissance Florence, » *Journal of European Economic History,* I (1972), 418-33, and, for further bibliography, p. 418, n. 1. The *abbaco* books have only recently been introduced into art historical literature: M. Baxandall, *Painting and Experience in Fifteenth Century Italy* (Oxford 1972), pp. 86-108; D. F. Zervas, « The *Trattato dell'Abbaco* and Andrea Pisano's Design for the Florentine Baptistery Door, » *Renaissance Quarterly,* XXVIII (1975), 483-503. For the possible influence of Piero's *Libellus de quinque corporibus regularibus* on the works of Bronzino, particularly on the Havemeyer portrait, C. H. Smyth, *Bronzino as Draughtsman* (Locust Valley, N. Y. 1971), pp. 83-85.

[35] *Scritti di Leonardo Pisano, matematico del secolo decimoterzo,* ed. B. Boncompagni, 2 vols. (Rome 1857-62).

[36] E. Colerus, *Piccola storia della matematica da Pitagora a Hilbert* (Turin 1962), p. 135; M. Cantor, *Vorlesungen über Geschichte der Mathematik,* 2nd ed., II (Leipzig 1892), 5.

[37] See below Chapter II, n. 38.

it seems from family chronicles, fell between nine and twelve years [38]. Once a geometrical section was commonly included in the *abbaco,* the texts became more systematic and purposeful, and the books then constituted an indispensable instrument in the education of aspiring artisans, artists, architects, and engineers.

The unpublished codex Ashburnham 359 in the Laurentian Library is representative of the whole genre of *abbachi.* Written by a Florentine around 1475, it begins, « *Inchomincio uno trattato d'abacho nel quale diamostro tutto quello che so partiene allo merchantio* » [39]. The introduction then teaches the multiplication tables, and succeeding chapters treat fractions, proportions, division of profit, methods of bartering, monetary exchange between various cities in Italy, and calculations with Florentine weights, measures, and currency. The « *regole delle tre cose* », or the « rule of three », the basis for most commercial transactions, formed one of the most important chapters of the *trattati d'abbaco.* A simple exercise of this kind asks, if five braccia of wool cost seven lire, what will three cost? Here the several elements are reduced to the simple proportion, 5 : 7 = 3 : x. In the Ashburnham *Abbaco* 359, however, the « rule of three » exercises develop into abstract proportional formulas which go beyond the requirements of pure mercantile exchange and become useful proportional tools for builders and architects. The anonymous author of Ashburnham 359 states the matter thus:

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[38] For example, we learn from Bernardo Machiavelli's *Libro di ricordi,* ed. C. Olschki (Florence 1954), pp. 31, 45, 103, 138, that his son Nicolò (b. 1469) began frequenting a « *maestro Matteo, maestro di grammatica,* » at age seven, in 1476. In 1479, at age ten, he began studying with Piero Maria, « *maestro d'abaco,* » and at age twelve, in 1481, presumably having completed the *scuola dell'abbaco,* he began the study of the Latin classics.

[39] Florence, Bibl. Medicea Laurenziana, MS. Ashb. 359, fol. 10r.

> Dobbiamo sapere che quando sono de lo più numerj de quali il primo sia al sechondo chome el sechondo al terzo & chome el terzo al quarto & chosi seguendo quelli numerj si dichono nella proportione chontinua e similmente quando sono in quantita e quando sono el primo al sechondo chome el sechondo al terzo & chome el terzo al quarto allora quelli numerj si dichono in proportione chontinua di chome siano nelle proportione chontinua e non chontinua [40].

Two examples are given. First, 8 : 12 : 18 : 27, which can be reduced to the proportion 2 : 3, 2 : 3, 2 : 3 (or fifths in musical consonances). And second, 12 : 18 : 24 : 36, reducible to 2 : 3, 3 : 4, 2 : 3 (in musical terms, a fifth, a fourth, and a fifth). The last chapter of the Ashburnham *Abbaco* 359, chapter 24, is devoted to geometrical exercises in the measurement of surfaces and volumes of solids [41]. Various abstract polygons are described and explained, but the most immediately geometrical exercises are practical ones dealing with such problems as the measurement of the contents of a barrel or a conical pile of grain, of the heights of walls, towers, and trees, and of long stretches over a land surface.

A survey of geometrical problems in a number of *abbaco* books shows that while they are suited to the needs of future merchants, they also provide, to a large extent, what future artisans, artists, architects, and engineers needed to learn. The Ashburnham *Abbaco* 359, for instance, also teaches the determination of the height of a column and the depth of a well, and it concludes with the measurement of the height of the

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[40] Ashb. 359, fol. 47r.

[41] Ashb. 359, fol. 142r: « *Nel 29° e ultimo chapitolo di questo trattato si dimostra chon brevita el modo del misurare le superficie chorpi e dalchuna altre cagionj piacevoli.* »

« *palagio de signori* », the Palazzo Vecchio in Florence [42]. Ashburnham *Abbaco* 518 demonstrates how to measure variously shaped tracts of land, pavilions, round and cubic stones, as well as the diameter of a column, round towers and walled fortresses. It also includes problems treating the measurement of unbroken walls and of walls pierced with arches, the painting of palace façades, the roofing of a house with tiles, the facing of the four sides of a house with stone, and the paving of piazzas with brick [43].

Another *abbaco* text, Ashburnham 356, instructs in measuring round vases [44], and Ashburnham *Abbaco* 1662 explains the measurement of a chalice [45]. Ashburnham 1379 shows how to measure columns and how to lay the brick pavement of the floor of a large hall [46]. Pier Maria Calandri's *Tractato d'abbacho* measures, among other things, a column, a chalice, wells, stones, and « *la palla di Sancta Liperata* », that which

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[42] Ashb. 359, fol. 206r: « *... et volessi misurare quanto e alto el palagio de signorj.* » The person taking the measurements stands on the corner of via Vaccherecia across from the palace and works out the solution using similar triangles. (Cf. also MS. Acq. e doni 154, fol. 224v.) Through geometrical problems in measuring buildings, like this one and the following one by Francesco di Giorgio, together with a study of optics treatises, the genesis of Brunelleschi's invention might be traced. FRANCESCO DI GIORGIO MARTINI, *La praticha di gieometria,* ed. G. Arrighi (Florence 1970), p. 21: « *Se per uno specchio l'alteza d'una torre saper vorremo, pigla lo specchio e ppollo in nel piano apresso alla torre, e ppoj ti muove innanzi e 'ndirietro tanto che ttu dirittamente veggha el' A in nel B, cioè la cima de la torre en el mezzo de lo specchio. E guarda quant'è la propotione de la distantia che è dal mezzo de lo specchio a' tuoj piedj e ll'altezza che è dal piè all'occhio tuo, cioè dal BC al CD, chosj hè dal BE a l'AE chome el BC e 'l mezo dello specchio a ppiè de la torre si metta. Esse altrjmente fusse, la proportione dal BA al CD altrjmentj sarebbe del piano a la torre. Unde mixura lo piano e propotionalo chome di sopra hera la torre.* » Cf. BOSKOVITS, « Prospettiva, » VIII (1962), 244; H. SIEBENHÜNER, « Zur Entwicklung der Theorie der Renaissance-Perspektive, » *Kunstchronik,* VII (1954), 129-30.

[43] Florence, Bibl. Medicea Laurenziana, MS. Ashb. 518, fols. 181r-207r.

[44] Florence, Bibl. Medicea Laurenziana, MS. Ashb. 356, fol. 23r.

[45] Florence, Bibl. Medicea Laurenziana, MS. Ashb. 1662, fol. 110v.

[46] Florence, Bibl. Medicea Laurenziana, MS. Ashb. 1379, fols. 87r-94r.

crowned the lantern of S. Reparata, the Florentine *Duomo*, until 1601 (Figs. 1, 2, 3)[47].

The *abbaco* books of the fourteenth and fifteenth centuries have only begun to be surveyed with an eye to gauging their rôle in the education of artists and artisans. But it is evident from the geometrical problems outlined above that these texts, the basis of the curriculum of a *scuola dell'abbaco,* were suited to teach beginning architects, engineers, painters, sculptors, goldsmiths, wood carvers, and intarsia designers. Vasari, in fact, relates that in his first months of study, Leonardo so progressed « *nell'abbaco* » that he confounded his master with the difficulty of the questions he raised[48].

In a way Piero's *Trattato d'abaco* represents the culmination of this genre of text book in the fifteenth century. Piero's introduction to his *Abaco* relates that he was asked to write « *alcune cose de abaco necesarie a' mercatanti* »[49]. Beginning

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[47] P. M. CALANDRI, *Tractato d'abbacho,* ed. G. Arrighi (Pisa 1974), fol. 219v. The diameter of the sphere crowning the lantern of the Duomo in Calandri's exercise is 4 braccia, which corresponds to the true measurement of Verrocchio's « *palla.* » C. GUASTI, *La cupola di Santa Maria del Fiore* (Florence 1857), p. 113: « *a misura di braccia quattro, meno l'altezza d'uno dito grosso.* »

[48] VASARI-MILANESI, IV, 18. Manetti (above n. 9), n. 52, relates concerning Brunelleschi: « *Nella sua tenera età, Filippo apparò a leggere ed a scrivere e l'abaco, come s'usa per gli uomini da bene e per la maggiore parte fare a Firenze.* » Similarly, concerning Bramante, Vasari (-Milanesi, IV, 147) wrote: « *Oltre il leggere e lo scrivere si esercitò grandemente nello abbaco.* »

[49] *Trattato d'abaco,* fol. 3r. Although Piero did not name his patron, it was one from whom « *i preghi suoi me sono commandamenti* » (fol. 3r). The *stemma* on the opening page of the manuscript, however, has been identified as that of the Picchi family, relatives of Piero from Borgo San Sepolcro (Ministero della Pubblica Istruzione. Indici e cataloghi, VIII: *I Codici Ashburnhamiani della Biblioteca Medicea Laurenziana di Firenze,* I, fasc. 6, ed. E. Rostagno and T. Lodi [Rome 1948], 462 ff.). BATTISTI (*Piero della Francesca,* I, 281), rightly dates the treatise to the years around 1450. For Piero's relationship to various members of the Picchi family: Battisti, II, index, s.v. « Picchi, famiglia di Sansepolcro. »

Although nothing is known of Piero's schooling, he came from a mer-

with fractions and the « rule of three », Piero moves quickly to advanced algebraic problems which, though they are still dressed in mercantile garb, transcend the needs of commercial transactions. The algebraic methods of computation to which Piero aspired were, we will see, innovative in the fifteenth century. Only in the sixteenth century were they finally perfected and brought to their logical conclusion. In a like manner, the geometry section of Piero's *Abaco* is more advanced than those of books similar to his. With two exceptions, there are no practical problems of measurement at all. Instead Piero deals exclusively with the measurement of abstract polygons and polyhedra. He divides his treatment of geometry into two sections, one on plane and the other on solid geometry. The first problems in plane geometry lay the foundation for the subsequent stereometrical demonstrations. Then, following the pattern of Euclid's *Elements,* Piero teaches the measurement of the five regular and other irregular solids inscribed in a sphere.

Although in the text of his *Abaco* Piero never indicates that he considers his treatise an advanced work, a statement in Alberti's *De' ludi matematici* shows that Piero's theme is more difficult than that which Alberti himself treats. « *Potrei in simili cose molto estendermi, ma queste per ora bastino. Se altro mi chiederete lo farò volentieri. Le misure de' corpi come sono colonne tonde, quadre, e auzze, di più faccie, speriche e simili*

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cantile family. L. COLESCHI, *Storia della città di Sansepolcro* (Città di Castello 1886), p. 262, describes Piero's father Benedetto as a « *mercante di panni di lana e calcolatore* »; see also BATTISTI I, 31. For a notice of Piero's activity as « *ingegnere idraulico,* » see G. SACCHETTI, *Sansepolcro, Alta Valle Tiberina* (Sansepolcro 1888), p. 78.

For a device related to that on fol. 3r of Piero's *Trattato d'abaco,* see P. M. CALANDRI'S *Tractato d'abbacho* (n. 47). Piero's scroll reads PI ... TEXT ... SI. The three scrolls in Calandri's text (fols. 3r, 9r, and 10r), read respectively: TEXT ... EII ... PSQ; TEXTE ... IP ... S .. V; TEXT ... EII. TEXT may imply *textere,* 'to weave' and relate to the wool guild.

corpi, e le tenute de' vasi e simili sono materie più aspre a trattare pur quando a voi dilettasse potrò ricorvele » [50]. Alberti's *Elementi di pittura* testifies, however, that exactly such studies dealing with multi-sided polygons constituted the basis of many artistic exercises for painters. Moreover, the complexities of Piero's particular problems ensured their usefulness for architects, engineers, sculptors, and intarsia designers. In fact, we will see that the geometry section of Piero's *Trattato d'abaco* is closely related to his *Prospectiva pingendi,* and that it forms the basis of the *Libellus de quinque corporibus regularibus.*

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Unlike his *Abaco,* Piero's book on the five regular bodies (the tetrahedron, cube, octahedron, icosahedron and dodecahedron) did not belong to a pre-existing genre. Instead, like the *Prospectiva pingendi,* it is original in conception and content. The *Libellus* presents for the first time a theme treated in many artists' manuals from the sixteenth century onwards. Such manuals aimed, at least in part, to teach the construction and perspectival drawing of the regular bodies.

The inspiration for Piero's treatise is found in the final books of Euclid's *Elements* — Books XIII, XIV, and XV — which culminate in the description, the construction, and the analysis of the relationships existing among the five regular polyhedra, the only equilateral, equiangular geometrical solids which can be comprehended in a sphere [51]. In Plato's *Timaeus* the five regular bodies were ascribed a cosmological significance as the symbols of the elements. The cube was associated

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[50] « Ludi, » p. 439.

[51] See *The Thirteen Books of Euclid's Elements,* ed. T. L. Heath, 2nd ed. rev. (New York 1956), III, for books XIII and XIV and a discussion of Book XV. Books XIV and XV, now known not to be the work of Euclid, still enjoyed his authority in the fifteenth century. For Books XIV and XV in the Campanus translation, which Piero used, see *Euclidis Megarensis philosophi acutissimi ... Opera,* ed. L. Pacioli (Venice 1509).

with earth, the octahedron with air, the pyramid with fire, the icosahedron with water. Finally the dodecahedron, which approaches most closely to the sphere, was associated with the cosmos for God used it « for arranging the constellations of the whole universe » [52].

The topic of the regular bodies was thus a fitting one for a Renaissance *trattatista* to choose from a philosophical as well as from an artistic and mathematical point of view. The accurate measurement of volumes and surfaces furnishes an exercise in the elements of sound painting, and, at the same time, it necessitates a through knowledge of mathematics. Although Piero does not mention the Platonic aspect of the regular bodies, it was of current interest, and he doubtlessly knew it well, for the cosmic, platonic implications of the regular bodies are expressly outlined in the penultimate proposition of Book XV of Euclid's *Elements* [53].

Piero dedicated the manuscript of his *Libellus de quinque corporibus regularibus* to Guidobaldo da Montefeltro, the son of Federigo da Montefeltro, Duke of Urbino, for whom Piero had earlier written his *Prospectiva pingendi.* In his ded-

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[52] PLATO, *Timaeus,* transl. with an introd. by H.D.P. Lee (Baltimore: Penguin, 1965), p. 75.

[53] EUCLID, *Opera,* ed. Pacioli, Book XV, prop. 12, fol. 144r-v. Other treatises on the five regular bodies are: (1) Triest, Biblioteca Civica « Attilio Hortis », R. P. MS. 2-33, discussed by G. ARRIGHI in FRANCESCO DI GIORGIO MARTINI, *La praticha di gieometria,* pp. 7-12, and (2) Regiomontanus (= JOHANN MÜLLER of KÖNIGSBERG), « De quinque corporibus aequilateris, quae vulgò regularia nuncupantur, quae videlicet eorum locum impleant naturalem & quae non, contra commentatorem Aristotelis, Averroem, » listed in J. G. DOPPELMAYR, *Historische Nachricht von den Nürnbergischen Mathematicis und Künstlern* (Nuremberg 1730), p. 19. Regiomontanus's treatise no longer exists,although there is a list of its chapters in a copy of another work, his « Commensurator. » See W. BLASCHKE and G. SCHOPPE, *Regiomontanus: Commensurator* (Wiesbaden 1956), pp. 474-75. The chapter headings are concerned with the construction of all five bodies, the measurement of some of them, and the derivation of one body from another.

ication Piero asks that the *Libellus* be placed with his book on perspective in the Urbino Library [54]. In the event, Piero's wishes were carried out and the books were shelved together. The proof is that, since Piero's perspective treatise lacked a title, Veterano's fifteenth-century catalogue entry for the manuscripts in the Urbino library records both works together as one: « 273. *Petri Burgensis pictoris Libellus de quinque corporibus regularibus, ad illustrissimum ducem Federicum et Guidonem filium* » [55].

The *Libellus* is divided into four parts. The first treats plane geometrical figures; the second, solid bodies inscribed in a sphere; the third, problems of regular bodies placed within one another and exercises concerning spheres; the fourth, irregular bodies. That Piero intended the book to stand « *penes aliud nostrum de Prospectiva opusculum, quod superioribus annis edidimus* » [56] indicates that he felt strongly the continuity of content between the *Libellus* and the *Prospectiva pingendi*. And, as we will see, that the topic of the regular and irregular bodies became an inevitable component of the exercises in later perspective manuals supports this conclusion.

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[54] *De corporibus regularibus,* p. 488.

[55] C. GUASTI, « Inventario della libreria urbinate compilato nel secolo XV da Federigo Veterano, bibliotecario di Federigo I da Montefeltro, Duca d'Urbino, » *Giornale storico degli archivi toscani,* VII (1863), 55.

[56] See above n. 54.

II.

The « Trattato d'abaco »: Algebra and Geometry

1. *The Provenance of the « Abaco » manuscript and its algebraic content*

Piero della Francesca's *Trattato d'abaco,* like other *abbaco* books of its kind, is a handbook of arithmetic, algebra, and geometry, written as a guide to mercantile mathematics. To this fact its opening lines testify explicitly.

> Esendo io pregato de dovere scrivere alcune cose de abaco necesarie a' mercatanti, da tale che i preghi suoi me sono commandamenti, non conmo presuntuoso ma per ubidire me sforçarò con l'aiutorio de Dio, in parte sactisfare l'animo suo, cioè scrivendo alcune raigioni mercantesche conmo baracti, merriti e compagnie; cominciando a la regula de le tre cose seguendo positioni et, se a Dio piacerà, alcune cose de algebra [1].

The *stemma* on folio 1r identifies its commissioner as a member of the Pichi family of Borgo Sansepolcro [2], a merchant no doubt, but one interested in the discipline of mathematics

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[1] PIERO, *Trattato d'abaco,* ed. G. Arrighi (Pisa 1970), fol. 3r.
[2] See above, Chapter I, n. 49.

for its own sake. Indeed Piero's exercises extend in complexity far beyond the needs of commerce.

Unrecorded and unknown until the twentieth century, the manuscript of Piero's *Trattato d'abaco* was identified by Girolamo Mancini in the Biblioteca Laurenziana in Florence around 1917 [3]. Recognizing the same hand in the pen drawings as in those of Piero's *Prospectiva pingendi* (Ambrosiana, C 307) and his *Libellus de quinque corporibus regularibus* (Vat. Urb. lat. 632), he also identified the script as Piero's by comparing it to the autograph Parma manuscript of the *Prospectiva pingendi* (Ms. 1576). Piero's *Abaco* manuscript derived from the collection of Lord Ashburnham in London, much of which had been bought from the Italian manuscript collector and mathematical historian, Guglielmo Libri. Thus Mancini suspected that Piero's manuscript once formed part of the Libri collection. Unbeknownst to Mancini, the proof of his well-founded suspicion lies in Libri's *Histoire de la mathématique en Italie* [4].

In the third volume of his *Histoire,* published in 1840, Libri transcribed a large part of the algebra section of Piero's *Trattato d'abaco* (folios 62r-79v) [5], describing it as an anonymous work. « *On trouvera dans cette note des extraits de deux anciens manuscrits d'algèbre qui m'appartiennent, et qui prouvent que depuis long-temps les géomètres avaient tenté de résoudre les équations des degrés supérieurs au second* » [6]. Libri

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[3] G. VASARI, *Vite cinque annotate,* ed. G. Mancini (Florence 1917), pp. 210-14.

[4] *Histoire des sciences mathématiques en Italie depuis la Renaissance des lettres jusqu'à la fin du dix-septième siècle,* 4 vols. (Paris 1838-41). For Libri's somewhat unorthodox methods of manuscript collecting, see L. DESLISLE, *Les manuscrits des fonds Libri et Barrois à la Bibliothèque Nationale. Extrait du catalogue des ces manuscrits* (Paris 1888). See also below, Appendix I, n. 3.

[5] *Histoire,* III (1840), 302-49.

[6] Ibid., p. 302.

dated the first manuscript, which was in fact Piero's, to the fourteenth century and located its origin in Tuscany. His dating is incorrect, for the first manuscript was the Ashburnham treatise, not an earlier version of the work. One may be certain of this on several counts. The marginal notes in Ashburnham 280, written in a sixteenth-century script, are reprinted in Libri's *Histoire* [7]. The penultimate problem in Libri's excerpt leads into Piero's geometrical section on regular bodies, in the *Abaco,* beginning on the next folio (80r). And finally, the same exercise recurs, in a Latin translation, in Piero's *Libellus de quinque corporibus regularibus* [8].

Libri's publication of Piero's work not only establishes the provenance of the manuscript. It also enlightens us concerning one branch of mathematics that occupied Piero, and it leads us to his medieval source. Libri considered these chapters in Piero's treatise important because they substantiated his belief that Italian mathematicians, since Leonardo Pisano, called Fibonacci, in the thirteenth century, had been attempting to solve algebraic equations higher than the second degree.

The solution of the cubic equation constituted the first forward step in mathematics since the inventions of the Arabs, and it took place in Florence in the first half of the sixteenth century. The discovery of the solution was probably due to Niccolò Tartaglia. It was then published by Gerolamo Cardano [9]. In any event, many mathematicians of the fifteenth and

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[7] See Ministero della Pubblica Istruzione, Indici e cataloghi, VIII: *I codici Ashburnhamiani della Biblioteca Medicea Laurenziana di Firenze,* I, fasc. 6, ed. E. Rostagno and T. Lodi (Rome 1948), 462, where the marginalia are identified, « *di mano del sec. XVI.* » Libri reprints in italics the marginal note on fol. 64r of the *Abaco* (p. 309). Moreover, the additional solution found in the margin of fol. 76v of the *Abaco* is reproduced by Libri (p. 340).

[8] *L'opera « De corporibus regularibus, »* ed. G. Mancini (Rome 1916), Part I, no. 4 (pp. 491-92).

[9] A concise explanation of the cubic equation and its historical im-

sixteenth centuries worked simultaneously to solve equations of the third, fourth, and fifth degrees. The ultimate solution of the cubic equation opened the way to modern algebra and modern algebraic equations. Piero's work in the *Trattato d'abaco* must be seen as a contribution to this process of discovery, and thus Libri's publication illuminates the breadth, novelty, and experimental character of Piero's mathematics.

Algebraic studies were first introduced into Italy by Leonardo Pisano, or Fibonacci, who codified the body of rules he learned in North Africa in his *Liber abaci* of 1202 [10]. This work remained the fundamental mathematical text until the sixteenth century. Not only did *abbaco* writers refer to Fibonacci time and again, but his work provided the source for Piero's investigations.

Book XV of Fibonacci's *Liber abbaci* treats rules in geometry pertaining to algebra. The section most significant for Piero, entitled « De solutione quarumdam questionum secundum modum algebre et almuchabale, scilicet ad proportionem et restaurationem » [11], instructs in the six forms of algebraic equations. Of these, Fibonacci writes, three are simple and three composite. The three simple equations are described thus:

> Primus quidem modus est, quando quadratus, qui census dicitur, equatur radicibus. Secundus quando census equatur numero; tertius quando radix equatur numero [12].

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portance is given in E. COLERUS, *Piccola storia della matematica* (Turin 1962), pp. 153-58; also: G. LIBRI, *Histoire,* III, 148-81; L. OLSCHKI, *Geschichte der neusprachlichen wissenschaftlichen Literatur,* III (1927; rpt. Vaduz 1965), 87-101; R. TATON, ed., *The Beginnings of Modern Science from 1450 to 1800* (London 1964), pp. 35-42; H. G. ZEUTHEN, *Geschichte der Mathematik im 16. und 17. Jahrhundert* (1903; facsimile rpt. New York 1966), pp. 81-93.

[10] L. PISANO, *Liber abbaci,* vol. I of *Scritti di Leonardo Pisano, matematico del secolo decimoterzo,* ed. B. Boncompagni (Rome 1857).

[11] PISANO, *Liber abbaci,* pp. 406 ff.

[12] Ibid., pp. 406-7.

That is, in the first case, the square of the unknown equals a square root; in the second, the square of the unknown equals a whole number; in the third, the square root of the unknown equals a whole number. Then Fibonacci describes the three composite equations.

> Primus enim modus est, quando census et radices equantur numero. Secondus, quando radices et numerus equantur censui, tercius modus est, quando census et numerus equantur radicibus [13].

In other words, in the first composite equation, the unknown squared plus a square root equal a whole number; in the second, the square root plus a whole number equal an unknown squared; and in the third, the unknown squared plus a whole number equal a square root.

Piero begins his algebraic chapters (fol. 24r) in a similar manner. He defines square roots (« radici »), and then continues to describe the simple and composite forms of equations.

> E, de queste, fa algibra 6 regule, tre semplici et tre composte. Le tre semplici sono quando nelle quistioni arismetrice o geumetrice se trova la cosa o vero radici equale al numero, o vero i censi equali a le cose, o vero il censo equale al numero. Però, quando le cose sono equali al numero, se dèi partire il numero per le cose e quello che ne vene vale la cosa. Et quando i censi sono equali a le cose, se dèi partire le cose per li censi e quello che ne vene vale la cosa. Et quando i censi sono equali al numero, se dèi partire il numero per li censi et la radici de quello che ne vene vale la cosa [14].

Composite equations are defined in the following way:

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[13] Ibid., p. 407.
[14] PIERO, *Trattato d'abaco,* fol. 24r.

> Et i composti sono quando i censi e le cose sono equali a li numeri, et quando i censi e i numeri sono equali a le cose, et quando il censo equale a le cose e al numero ... [15].

To Fibonacci's six forms, Piero adds an additional fifty-five possibilities that employ cubic equations, biquadratic equations, and equations of the fifth degree [16]. Nearly half of these fifty-five rules are fully illustrated with problems, and these are precisely the pages that Libri transcribed in his *Histoire de la mathématique.* One example of Piero's more complex algebraic equations shows how he elaborates the six basic types.

> Quando le cose e i censi et i cubi e censi de censi sono equali al numero, se dèi partire nei censi di censi e poi partire le cose per il cubi et, quello che ne vene, recare a radici et ponere sopra del numero; et la radici de la radici di quella somma meno la radici de le cose che ne venne partite per li cubi, tanto vale la cosa [17].

Here we meet with a biquadratic equation — one including a simple unknown, the square of an unknown, the cube of an unknown, and an unknown raised to the fourth power. The problem that illustrates this rule concerns the interest rate to be paid on 100 *Libre* over four years.

Just over a half a century after Libri's volume appeared, Piero's problems, as Libri had printed them, were closely ana-

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[15] Ibid.

[16] Ibid., fols. 30r-78v. For an explanation of Piero's equations see S. A. JAYAWARDENE, «'The Trattato d'abaco' of Piero della Francesca,» in *Cultural Aspects of the Italian Renaissance, Essays in Honour of Paul Oskar Kristeller,* ed. C. Clough (Manchester 1976), pp. 233-34, a work published after the present study was completed.

[17] Ibid., fol. 67v.

lyzed by Moritz Cantor in his unsurpassed *Vorlesungen über Geschichte der Mathematik* [18]. Even without the manuscript itself, Cantor was so impressed with Libri's excerpts that he evaluated the work in the section of his book devoted to the development of mercantile mathematics. On the one hand, he points out some of Piero's false steps in attempting to solve cubic and other higher equations. But, on the other, Cantor maintained that, though in the last analysis unsuccessful, his anonymous algebraicist was an uncommonly good arithmetician, indeed a very gifted mathematician, and he suspected, from the penultimate problem on triangles, that his man was equally talented in geometry [19]. Knowing only the single geometrical exercise Libri had published, Cantor speculated no further. It testifies to the novelty of Piero's work that Cantor devoted far more attention to Piero than to any other contemporary algebraicist.

2. *The Contents of the geometrical sections of the « Abaco »*

We have seen that Piero's algebraic studies have their own significance in the history of mathematics. His geometry chapters, as Cantor suspected, reveal a no less gifted mind. Moreover, Piero's geometry lessons furnish a key to understanding the entirety of his treatises. By carefully following the geometrical exercises in the *Trattato d'abaco*, by isolating certain themes running through the entire work, and by determining their sources, the interrelatedness of the *Trattato d'abaco,* the *Libellus de quinque corporibus regularibus,* and the *Prospectiva pingendi* may be seen. Thus a continuity exists in Piero's written works, for all three treatises stem from related

[18] 2nd ed., II (Leipzig 1892), 144-50.
[19] Ibid., p. 149.

interests. Piero's mathematics and his art need not be counted as separate phenomena; they are components of the same intellectual enterprise, carried forward by a single individual.

Earlier we examined some of the geometric themes which the normal *trattato d'abbaco* treated. The Florentine *abbaco* of Pier Maria Calandri, for instance, defines a solid simply as « *quello che à lunghezza, larghezza et altezza* », and it measures objects like chests, wells, vats, columns, towers, trees, and walls [20]. A Neapolitan *abbaco* of circa 1478 divides all solids into two basic categories, columnar and pyramidal. Here, too, the illustrations are drawn from objects such as wells, barrels, and chests [21]. In contrast, the material which Piero della Francesca presents in the geometry chapters of his *Trattato d'abaco* distinguishes the work from its contemporaries. Piero treats almost exclusively the measurement, not of everyday objects, but of abstract geometrical polygons and polyhedra. In his treatment of these two-dimensional figures and three-dimensional solids, two central themes stand out — the golden section and the five regular bodies. The golden section exercises, like Piero's work in algebra, find their mathematical foundation in the work of Fibonacci, as well as in Euclid's *Elements*. The later exercises that deal with the five regular bodies are a direct consequence of Piero's Euclidean studies.

The geometrical sections of Piero's *Trattato d'abaco* divide into two parts. The first (fols. 80r-104v) treats plane geometry — the measurement of two-dimensional polygons. The second (fols. 105r-120v) contains exercises in solid geometry — the

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[20] P. M. CALANDRI, *Tractato d'abbacho*, ed. G. Arrighi (Pisa 1974), fols. 216v, 217r-223v.

[21] P. P. MUSCARELLO, *Algorismus: Trattato di aritmetica pratica e mercantile del secolo XV* (Verona 1972), I, fols. 86v-87r: « *Item corpi si sono di 2 manere principalminte, prima si è colonna, 2a è pirramide. Colonne sono como casse, arche di grano, pozi, botte, tine. Pirramide sono como tramoggia, monti di grano et omne figura ampia di sopto et apontata di sopra* ».

measurement of three dimensional solids (often including three-dimensional bodies placed in a sphere). The rules for the measurement of triangles, squares, rectangles, parallelograms, pentagons, hexagons, octagons, and circles, taught on the first forty pages, establish the foundation needed to measure the succeeding solid geometrical bodies. Piero's definitions of the several polygons are remarkably clear, for instance, his definitions of the pentagon and hexagon (fols. 88v-91v).

> El pentagono è una figura de 5 lati equali et de 5 anguli equali, de la quale figura i lati suoi s'à da la corda de l'angulo pentagonico ... La figura de lo exagono è una figura de 6 lati equali a' semidiametro del circulo dove è descricta; e la superficie sua se devide in sei trianguli equilateri per li quali s'à la superficie sua mediante il catecto de uno de quelli trianguli.

Equally clear are the problems and solutions Piero devises to explicate each of these plane geometric figures. What is the area of the equilateral hexagon ABCDEF, Piero asks, on folio 91v, whose side is 6 braccia? By explaining the relationship of the equilateral hexagon to the equilateral triangle and by referring back to his earlier lessons on the triangle, Piero makes the solution readily comprehensible.

> Fa'così. Tu sai che se devide in sei trianguli; pigla uno de questi trianguli che sono tucti equilateri et trova il suo catecto per quella via che te fu mostro nelli trianguli, che troverai che sirà la radici de 27. Et tu ài che a montiplicare il catecto per la metà de la basa ne vene la superficie del triangulo, adunqua pigla la metà de le base che sono 6, sirano 3, et è ciascuna 6 bracci, sirano 18. Il quale 18 montiplica in sé farà 324, et questo montiplica per lo catecto ch'è 27 fa 8748; et la radici de 8748 è la superficie del dicto exagono ABCDEF, che è 6 bracci per facia.

Piero's precision and thoroughness in dealing with these abstract figures is striking when compared with the definitions

of two-dimensional figures found in contemporary *abbachi*, where often the simplest definitions of very elementary forms sufficed. In the Neapolitan *abbaco* of 1478, fol. 86v, we read, « *Item pentangoli che ànno 5 lati e 5 angulli, exangulli à 6 anguli e lati, multilateri ànno multilati e multi anguli* ». And in Pier Maria Calandri's contemporary Florentine *Tractato*, triangles, rectangles, and pentagons are described in the following manner: « *Le figure di tre linee rette si dichono triangholi, le figure di quattro linee rette si chiamano quadrangoli, le figure di 5 lati si chiamano pentaghoni...* » (fol. 209v).

Folios 105r-120r of the *Trattato d'abaco* present exercises in measuring solid geometric forms and in demonstrating their proportional relationship to each other and to the sphere. Piero treats the five Platonic-Euclidean bodies and the sphere, as well as irregular pyramids and solids of eight and fourteen faces, similarly placed in a sphere. The exercises are far more complex than the simple polygons in Alberti's *Elementi di pittura* and they, too, go far beyond the usual practical exercises of measurement in contemporary *abbaco* handbooks.

3. *The Golden Section, the Five Regular Bodies, and Piero's « Abaco »*

The golden section first occurs in the plane geometry exercises of Piero's *Abaco* in the discussion of the pentagon; it recurs in the solid geometry section when Piero demonstrates the measurement of a dodecahedron (composed of twelve pentagons) in a sphere. Despite many specialized studies, there exists no general agreement as to how widespread the use of this proportion was among Renaissance artists [22]. Nonetheless,

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[22] P. H. Scholfield, *The Theory of Proportion in Architecture* (Cambridge 1958), pp. 52-54, and bibliography, pp. 148-50; R. Wittkower,

Piero's use of it speaks for its knowledge in artistic circles, and the clarity of Piero's exposition must have fostered its further diffusion.

The peculiarity of the golden section is that, unlike most Renaissance proportions, it is incommensurable, that is geometrical, and cannot be expressed in terms of rational integers, that is of discrete whole numbers. To divide a line according to the golden section, or in its mean and extreme ratio, signifies that the resulting smaller section is in the same proportion to the larger section as the larger section is to the whole line. This can be reduced to a formula. Given the line ABC, where AB is the smaller section and BC the larger, AB : BC = BC : AC, or BC^2 = AB.AC. The proportion first appears on folio 88v, where the problem is to determine the side of a pentagon when the chord is 12 braccia. « *Tu dèi sapere* », Piero writes on folio 88v, «*che* 12 *se dèi devidere secondo la proportione avente il meçço et doi stremi, e la magiore parte è il lato del pentagono esendo* 12 *la corda de l'angulo pentagonico* ». In the result, the side of the pentagon (equal to the larger section of the chord) is $\sqrt{180}$-6. This type of application of algebra to geometrical principles (the process whereby Piero resolves the question) finds its roots in Fibonacci, whose work proved so decisive for Piero's algebra. Moreover, the division of the chord of a pentagon that equals 12 into its mean and extreme ratio is also found in Fibonacci.

Book VI of his *Practica geometriae* (1220), entitled « In dimensione corporum », treats the division of a line into its mean and extreme ratio[23]. The first of Fibonacci's two

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« The Changing Concept of Proportion, » *Daedalus* (Winter 1960), p. 205-09. Cf. G. De Angelis d'Ossat, « Enunciati euclidei e 'divina proportione' nell'architettura del primo Rinascimento, » in *Il mondo antico nel Rinascimento, Atti del V Convegno internazionale di studi sul Rinascimento, Firenze 1956* (Florence 1958), pp. 253-63.

[23] L. Pisano, *Practica geometriae,* vol. II of *Scritti di Leonardo Pisano, matematico del secolo decimoterzo,* ed. B. Boncompagni (Rome 1862),

demonstrations corresponds to Piero's, for he divides the chord of a pentagon that measures 12 into its mean and extreme [24]. Fibonacci earlier in Book VI had made explicit the relationship between the division of a line according to the golden section and the pentagon: « *In circulo descripto penthagono equilatero duorum eius angulorum corde sese invicem secent, utraque secari necesse est. Ad idem punctum proportione medij et extremorum maior utriusque pars equalis lateri penthagonico in omni circulo, cuius dyametrum rationali penthagoni equilateri latus inrationale est, cui nomen linea minor* » [25].

Book VIII of Fibonacci's *Practica geometriae,* « De quibusdam subtilitatibus geometricis », is devoted to problems of the pentagon and the closely related decagon [26]. Fibonacci demonstrates, for example, how to find the length of a side of a pentagon inscribed in a circle whose diameter is 10, the diameter of a circle inscribed within a pentagon whose side is 10, and the side of a pentagon whose area is 50 [27]. Similarly, Piero shows how to find the sides and area of a pentagon inscribed in a circle whose diameter is 12 [28].

Other relations to Fibonacci can be discovered in Book XV of Fibonacci's *Liber abbaci,* a work devoted to the rules of geometry that pertain to algebraic questions [29]. Here Fibonacci demonstrated at length the division of numbers into two parts, often working with the number 10 [30]. One exercise di-

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p. 196: « *Modus dividendi lineam media et extrema proportione.* »

[24] PISANO, *Practica geometriae,* pp. 196-97: « *Nam modus dividendi corda .be. media et extrema proportione est ut super quadratum eius, quod est .12.* »

[25] *Practica geometriae,* pp. 161-62.

[26] Ibid., pp. 207 ff.

[27] Ibid., pp. 207-16.

[28] PIERO, *Trattato d'abaco,* fol. 90r-v.

[29] PISANO, *Liber abbaci,* pp. 387-459, esp. 387: « *Incipit capitulum quintum decimum de regulis geometrie pertinentibus, et de questionibus aliebre et almuchabale.* »

[30] *Liber abbaci,* pp. 410-59.

vides 10 into two parts, so that the first part divided by the second, plus the second divided by the first, and the whole multiplied by the first, yields 34 [31]. It is noteworthy that Fibonacci's work includes three methods of resolving the division of 10 according to the golden section, so that the smaller section will equal $15 - \sqrt{125}$, and the larger $\sqrt{125} - 5$, and, he writes, there are still other ways to perform this operation. « *Possemus etiam in his aliis modis procedere; sed ista que diximus, sufficiant: et scis, secundum hanc divisionem,* 10 *divisa esse media et extrema proportione; quia est sicut* 10 *ad maiorem partem, ita maior pars ad minorem* » [32]. Subsequently, Fibonacci demonstrates that the resulting sections, $15 - \sqrt{125}$ and $\sqrt{125} - 5$, multiplied according to the golden section formula, yield 10, the length of the line to be divided. Thus the methods of division were painstakingly established for other readers to study and to acquire skills in performing.

Piero's exposition of the golden section is more exhaustive and it is, at least in part, based on Book XIII of Euclid's *Elements*. On folio 90r-v Piero demonstrates how to determine the length of the sides and area of a pentagon inscribed in a circle whose diameter is 12. According to Euclid, XIII, 8, writes Piero, the side of a hexagon joined to the side of a decagon inscribed in the same circle results in a line divided into its mean and extreme ratio; furthermore, according to Euclid XIII, 9, the side of a hexagon joined to the side of a decagon is equal to the side of a pentagon inscribed in the same circle; and, following Euclid, XIII, 10, in the chord of a pentagon divided into its mean and extreme, the larger section is equal to the side of a pentagon. He also knew from Euclid that the side of a hexagon equals half the diameter of the circle

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[31] Ibid., p. 419.

[32] Ibid., p. 438. Cfr. also Pacioli's exercises with the division of 10 into its mean and extreme in the *Divina proportione* (Venice 1509), Part I, Chapters 10-22.

in which it is inscribed. With the diameter of the circle as a known quantity, Piero could then proceed, basing his steps on the three propositions he had listed from Book XIII of Euclids's *Elements*. Applying algebraic methods to the geometrical principles of Euclid, Piero then arrived at the side of a pentagon by finding the side of a decagon inscribed in the same circle and adding to it the known side of a hexagon. And, he provides the formula for the golden section, « *però che sia una linea cusì devisa che la menore parte montiplicata in tucta la linea faccia tanto quanto la maggiore parte montiplicata in se medessima* » [33].

A third example of the golden section in Piero's *Abaco* is elucidated in his discussion of the dodecahedron inscribed in a sphere [34]. This, of course, derives from problems concerning the pentagon, for the dodecahedron is composed of twelve regular pentagonal faces. Piero attempts to find one side of the face of a dodecahedron inscribed in a sphere whose axis is $\sqrt{48}$.

> Tu dèi sapere che il lato del cubo descricto in una medessima spera deviso, secondo la proportione avente meççço e doi stremi, che la magiore parte è il lato del corpo de 12 baxe pentagonali descricto in una spera. Et ài, per la 13a del 13 de Euclide, che la posança del diametro de la spera è tripla a la posança del lato del cubo da quella contenuto [35].

Piero then proceeds to find the side of the cube, which, according to Euclid's formula, is one third the axis, or 4. The methods for dividing 10 into its mean and extreme had been taught, as we have seen, by Fibonacci in his *Liber abbaci* [36].

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[33] Piero, *Trattato d'abaco,* fol. 90v.
[34] Ibid., fol. 110r.
[35] Ibid.
[36] Pisano, *Liber abbaci,* pp. 410-59.

The methods then existed, and Piero only had to apply them to the division of 4 in order to determine the pentagonal side of a dodecahedron.

It is apparent that Piero was at home in Fibonacci's *Liber abbaci* and *Practica geometriae,* works which constituted the principal sources for late medieval and early Renaissance mathematics. The proportion that modern times calls « golden » and the Renaissance called « divine », Piero and Fibonacci leave without a name. Both refer to it, in Euclidean terms, more simply as the proportion for dividing a line in its mean and extreme ratio. Known since classical times, it was necessary to Euclid's construction of the pentagon and, consequently, also the dodecahedron. Fibonacci, and Piero in his wake, treat the proportion as one way of dividing a line, one way of several, and do not attribute the extraordinary characteristics to it that Luca Pacioli later does [37]. Nonetheless, Piero's students, as Fibonacci's before, whether they were merchants, mathematicians, architects, or artists, learned, if they followed their lessons diligently, the division of a line in its mean and extreme ratio, that is the golden section, which later excited such interest and acquired the connotations of divine.

Even though Piero does not explicitly cite Fibonacci in his *Abaco,* there is no doubt that Fibonacci provided one of the foundation stones of Piero's treatise. Piero's immediate sources are not to be found in earlier *abbaco* texts, where solid geometrical principles were generally applied to everyday objects, not to abstract geometrical forms. It is precisely these abstract geometrical figures and solids that most concern Piero, well-versed as he was in Euclid's *Elements.* The application of algebraic methods to geometrical forms can be traced, in part, back to Fibonacci. Though Piero's relationship to Fi-

[37] *Divina proportione,* Part I, Chapters 10-22.

bonacci has not been previously examined, the connection is, for a number of reasons, a logical one to make. In a very elementary sense Fibonacci's work was fundamental for the entire genre of *abbaco* textbooks. Contemporary *abbaco* writers frequently cite Fibonacci, referring to him as Leonardo Pisano, or, more familiarly, as « *L. P.* » Not only is Fibonacci's authority invoked, but a fifteenth-century treatise in applied arithmetic, « *tratto da libri di Leonardo pisano* », credited Fibonacci with the revival of mathematics in Tuscany. Its author reports that by around 1348, not so long after Fibonacci lived, there were enough flourishing *scuole d'abaco* in Florence to accommodate tens of hundreds of *fanciulli* [38].

The golden section is closely allied to the second theme that distinguishes Piero's *Trattato d'abaco,* the five regular bodies. They are the only five equal-faced, equiangular solids that can be comprehended in a sphere, the tetrahedron, the cube, the octahedron, the icosahedron, and the dodecahedron. The *tetrahedron* is a pyramid composed of four equal, equilateral, and hence equiangular faces. The *cube* has six square faces. The *octahedron,* a double pyramid on a square base, has eight equal and equilateral triangular faces. The last two

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[38] See the description of the *scuole dell'abbaco* in a treatise on practical arithmetic (« Trattato di praticha darismetica tratto da libri di lionardo pisano, » Siena, Bibl. Pubblica Comunale, Cod. L. IV. 21, fols. 127r-v) of 1460, in B. Boncompagni, *Intorno ad alcune opere di Leonardo Pisano* (Rome 1854), p. 251: « *Al fatto del chaso di L. p. Dicho che L. p. fu huomo sottilissimo in tutte dispute, et secondo che si truoua lui fu il primo, che ridusse allume questa praticha in toschana, che allora sandava per vie molte estrane, nientedimeno dassai tenpo inanzi allui in questa nostra città furono schuole dabacho, che circha al 1348. ò veduto Trattato che dice in firenze essere più di 10. centinaia di fanciulli alle schuole dellabacho, che pocho inanzi fu lionardo. E anchora chome si uede lonsegnare loro era a modo antichi et quasi al modo che oseruono di presente e vinitiani, che .è. maraviglia al sufficienti (sic) maestri vi sono stati, et sono chome e non anno ridotto in una facile praticha tutto.* »

regular bodies approach more nearly the form of the sphere. The *icosahedron* has twenty equilateral triangular faces, and though the *dodecahedron* has only twelve faces, they take the form of regular pentagons. The perfection of these forms exercises an inevitable fascination, and that there are invariably, forever and ever, five and only five such bodies, is one of those eternal truths that have recurrently struck men as marvellous. Plato singled out these exclusive, essential forms as his symbols for the four elements and the universe [39]. And Euclid made them the culmination of his compendium of solid geometry in Book XIII of the *Elements*. Moreover, the regular bodies form the subject of two further books of the *Elements,* XIV and XV, which, though now excluded from the Euclidean canon, were considered as an integral part of his work in the early Renaissance. Piero's *Abaco* exercises concerning the five regular bodies furnish the core of his later *Libellus de quinque corporibus regularibus* and bring us closer to his second basic source, Euclid's *Elements,* Books XIII-XV. (Here we must follow early Renaissance usage regarding Euclid's *oeuvre*.)

The proportion that divides a line in its mean and extreme ratio, the « golden section », is fundamental to Book XIII of Euclid's *Elements,* on the regular bodies, for it is the subject of the first six propositions, and it is necessary to the construction of the dodecahedron. Luca Pacioli made the connection between the regular bodies and the golden section explicit in the title to Book XIII of his 1509 edition of Euclid's *Opera,* « Liber tertiusdecimus Euclidis de admiranda vi linee secundum proportionem haventem medium duoque extrema divise et quinque corporum regularium formatione » [40]. With Piero the

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[39] See above, Chapter I, n. 52.

[40] *Euclidis Megarensis philosophi acutissimi ... Opera,* ed. L. Pacioli (Venice 1509), fol. 122r. Pacioli had also taught the proportional measurement of the regular bodies in Chapters 26-31 of the *Divina proportione.*

connection is left implicit, as, for instance, in his discussion of the pentagon and dodecahedron, which we examined above. Nonetheless, understanding regular bodies required understanding the division of lines into their mean and extreme ratios.

The second part of the geometry section of Piero's *Abaco* (fols. 105r-120r) treats regular and irregular bodies, their relationship to a sphere and to one another. In this section Piero's relationship to Euclid becomes evident in such references in his text as « *conmo per la penultima del primo de Euclide se demostra* ». Similar references, however, abound in Fibonacci, and we may not exclude *a priori* the possibility of an indirect transmission of Euclidean geometry via Fibonacci. Nonetheless, Piero's first-hand knowledge of the works of Euclid can be demonstrated by a close reading of his text.

Piero begins his solid geometry section, « *Il corpo à tre demensioni, cioè longheçça, largheçça e profondità; et sono de più ragioni, benché io non intenda dire se none de alcuni conmo è de' 4-base triangulare et de' cubi e de' tondi sperici* » [41]. Though Piero does not intend to discuss all solids, those with equal sides and faces will be included, and these are the five regular bodies. Piero first discusses the cube and the tetrahedron, including their interrelationships, their relations to the sphere, and some of their irregular derivatives. Only when he reaches the octahedron does he reveal his intention to discuss all five Platonic-Euclidean solids. « *De questo corpo* », he writes, « *facilemante s'àno le sue mesure, però non de' credecti dire niente et è remaso de rieto per questo; ma, perché ci sieno tucti 5 li corpi regulari non voglio lassare* » [42]. In the remaining exercises in solid geometry Piero redeems his prom-

[41] PIERO, *Trattato d'abaco,* fol. 105r.
[42] Ibid., fol. 114r.

ise. All five regular bodies are explained, the tetrahedron (fol. 105r-v), the cube (fol. 106r-v), the dodecahedron (fols. 110r-111v), the icosahedron (fols. 112r-113v), and the octahedron (fol. 114r).

Fibonacci's discussion of the regular bodies in his *Practica geometriae* is neither so extensive as Piero's nor as complete. Having listed only three of the five solids, Fibonacci directs his reader to Euclid, who teaches their construction in a sphere. « *Solida multarum basium sunt multimoda, ex quibus sunt solida VIII. basium, et XII. basium, et XX. basium equalium, que Euclides in xiiiº libro infra speram construere docet* »[43]. Save for a short excursus on the dodecahedron and icosahedron, Fibonacci is more concerned with the measurement of practical bodies[44].

For an extensive explanation of the regular bodies we are obliged to take Fibonacci's advice and turn to Euclid. There we find the evidence (especially in light of Piero's later treatise devoted to the regular bodies) that Piero, too, went back to the ancient source, for the theorems that lie at the heart of Piero's work are laid out consecutively in propositions 13 to 18 of the thirteenth book of Euclid's *Elements*.

Book XIII of Euclid's compendium instructs in the divi-

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[43] PISANO, *Practica geometriae*, p. 159.

[44] FIBONACCI writes (p. 158) that bodies are of many kinds such as solids, cones, pyramids, columns, and spheres: « *Corporum quidem genera sunt plura, ex quibus sunt hec: Solidi, Seratilia, Piramides, Columne, Spere, et eorum partes, nec non et corpora, que multorum basium nuncupantur, et describuntur circa speras: proprie quidem solidum est, quod latum longum et altum habet; et constat ex sex superficiebus: ut sunt tassilla: scrinea, cisterne et similia.* » In the solid geometry section of the *Liber abbaci* (pp. 403-05), Fibonacci deals with the problems of water displaced in a cistern into which pyramidal-shaped stones, columns, or spheres are placed Furthermore in finding the area of a square based pyramid the problem is couched in the familiar, applied mathematical terms of three men painting the pyramidal roof of a palace (p. 405).

sion of a line into the golden section (propositions 1-6). It also treats the construction of the regular bodies in a sphere, together with their proportional relationship to the sphere (propositions 13-17) and to one another (proposition 18). These propositions relate directly to the regular bodies discussed by Piero. Proposition 14, for instance, is required to solve Piero's problem with the octahedron (fol. 114r), « find the side of an octahedron comprehended in a sphere whose diameter is 10 ». Euclid's proposition establishes that the square on the diameter of the sphere (100) is double the square on the side of the octahedron (or $2x^2$). Hence $100 = 2x^2$, or the side of the octahedron equals $\sqrt{50}$. In addition, Euclid's proposition 15 is necessary to the solution of Piero's problem determining the side of a cube comprehended in a sphere whose diameter is 7 (fol. 107r). Knowing from Euclid that the square of the diameter (or 49) equals three times the square of the side of a cube ($3x^2$), Piero quickly demonstrates the answer.

For one more instance, Proposition 9 of Book XIV of the *Elements* is fundamental to Piero's finding the area of a dodecahedron whose side is 4 (fol. 110v), Piero writes: « *Tu ài per la 9a del 14° de Euclide, che li tre quarti del diametro del circulo dove è descricto il pentagono, montiplicato in cinque sexti de la linea che soctotende l'angulo pentagonico prova che fa la superficie del pentagono* ». There are many other such examples in Piero's text which relate directly to a manuscript source of Euclid's *Elements*.

In the problems concerning the dodecahedron and icosahedron (fols. 110r-113v), Piero utilized extensively Book XIV of the *Elements*, whose propositions 5, 6, 7, 8, 9, and 10 illustrate the relationship of these figures to each other as well as to the sphere that comprehends them.

The theme of the regular bodies reaches its culmination, of course, in the *Libellus de quinque corporibus regularibus*, and the following chapter will provide a detailed comparison of this manuscript with the *Abaco*. For the present the *Tratta-*

to d'abaco may be defined as Piero's initial (or at least early) foundation for his *Libellus*, a work even more clearly Euclidean in character.

4. *The « Abaco » and the « Prospectiva pingendi »*

To all appearances Piero's *Trattato d'abaco* is a work of pure mathematics. In it Piero never mentions the relevance of solid geometry to artists, yet it is precisely the measurement of the surfaces of geometrical bodies that is one of the primary concerns of the science of perspective. Of the three principal parts of painting, « *disegno, commensuratio et colorare* », Piero limits himself to the second in his *De prospectiva pingendi*: « *Commensuratio* », he writes, « *diciamo essere essi profili et contorni proportionalmente posti nei luoghi loro* », and this is also called « *prospectiva* » [45]. He will not treat color, but instead, « *tracteremo de quella parte che con line angoli et proportioni se po dimostrare, dicendo de puncti, linee, superficie et de corpi* » [46]. The first of the three books on perspective treats « *puncti, de linee et superficie piane* », the second, « *corpi chubi, de pilastri quadri, de colonne tonde et de più facce* », and the third, « *le teste et capitelli, base, torchi de più base et altri corpi diversamente posti* » [47]. The pure mathematical definitions given in an *abbaco* are then redefined for the artist. For the mathematician, Piero writes in the *Prospectiva pingendi*:

> Puncto è la cui parte non è, secondo i geumetri dicono essere inmaginativo; la linea dicono avere lunghezza senza latitudine.

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[45] *De prospectiva pingendi*, ed. G. Nicco Fasola (Florence 1942), p. 63.
[46] Ibid., p. 64.
[47] Ibid., p. 65.

And for the artist, he writes:

> Dirò adunqua puncto essere una cosa tanto picholina quanto è posibile ad ochio comprendere; la line dico essere extensione da uno puncto ad un altro, la cui larghezza è de simile natura che è il puncto. Superficie dico essere larghezza et longhezza compresa da le linee. Le superficie sono demolte ragioni, quale triangola, quale quadrangola, quale tetragona, quale pentagona, quale exagona, quale octagona, et quale de più et diverse facce, commo per figure ve se dimostrarà [48].

The introduction to Book II of the perspective treatise testifies most clearly to the relationship between the *De prospectiva pingendi* and the geometry of the *Abaco*.

> Corpo ha in sé tre dimensioni: longitudine, latitudine et altitudine; li termini suoi sono le superficie. I quali corpi sono de diverse forme, quale è corpo chubo, quale tetragono che non sono de equali lati, quale è tondo, quale laterato, quali piramide laterate, et quale di molti et diversi lati, sicommo ne le cose naturali et ancidentali se vede. De li quali in questo secondo [libro] intendo tractare de la loro degradationi, nelli termini posti da l'occhio socto angoli compresi, facendo de alcune superficie degradate nel primo lor base [49].

These are the solids, the cubes, pyramids, and spheres, the many-faced bodies that Piero had described earlier. For them he had invented mathematical exercises in measurement in the Ashburnham *Abaco*. Finally, the importance of these geometrical bodies for painting is attested to in the introduction to Book III of the *Prospectiva pingendi*: « *Et perché la pictura non è se non dimostrationi de superficie et de corpi degrada-*

[48] Ibid., pp. 65-66.
[49] Ibid., p. 100.

ti o acresciuti nel termine, posti secondo che le cose vere vedute da l'occhio socto diversi angoli s'apresentano nel dicto termine... »[50].

The mathematical rules for the correct, scientific measurement of regular and irregular geometrical solids demonstrated in the *Trattato d'abaco* were, in a way, preliminary and necessary to a thorough understanding of the *Prospectiva pingendi.* As Piero wrote, a painter needed to know the significance of lines and angles in order to understand perspective. Hence the study of regular solids, and their measurement and placement in a composition, was of interest to mathematicians and painters alike. The theorems that lie at the heart of Book XIII of Euclid's *Elements* instruct the mathematician in the relationship of one regular body to another and train the painter to see and reproduce the inanimate bodies found in nature. A few years later, Luca Pacioli's *Divina proportione* (in essence a treatise based on the Euclidean-Platonic five regular bodies) was dedicated equally to students of «*philosophia, prospectiva, pictura, sculptura, architectura, musica, e altre mathematice* »[51].

[50] Ibid., pp. 128-29.
[51] *Divina proportione,* title page.

III.

The «Libellus de quinque corporibus regularibus»: Regular and Irregular Bodies

The *Libellus de quinque corporibus regularibus* (Vat. Urb. lat. 632), dedicated to Guidobaldo da Montefeltro, Duke of Urbino, was composed in the decade between 1482 and Piero's death in 1492 [1]. The words of praise and appreciation which Piero lavishes on the family that so often sponsored him, in his dedication to Guidobaldo, are not entirely original for they are based on a Vitruvian formula. Nonetheless, the sentiments he expresses are an accurate reflection of Piero's close relations with the young duke and with his late father, Federigo da Montefeltro. Moreover, in his dedication to Guidobaldo, Piero specifies the primary innovation of his *Libellus* and indicates its connection to his earlier *De prospectiva pingendi.*

In writing to Guidobaldo that the lasting fame of many artists rested primarily on the men they served, Piero paraphrases the introduction to Book III of Vitruvius' *De architectura.*

> Inter antiquos pictores et statuarios, Guido princeps insignis, Policretum, Phidiam, Mironem, Praxitelem, Apellem, Lisippum,

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[1] *L'opera « De corporibus regularibus, »* ed. G. Mancini (Rome 1916). In 1482 Federigo da Montefeltro died, and his son Guidobaldo succeeded him as Duke of Urbino.

> ceterosque qui nobilitatem ex arte sunt consecuti, non ob aliud digniores fuisse, et apud suos maiorem gratiam, apud vero posteritatem memoriam et famam diuturniorem, Aristomene, Thasio, Polide, Chione, Pharaxe, Boeda, ceterisque, qui non minori artis studio, ingenio, solertia, et industria fuerunt, habuisse perhibent, nisi quod ii aut civitatibus magnis, aut regibus, aut principibus, virtutis experimentatae opera fecerunt [2].

Then, continuing his praise of the house of Montefeltro, Piero reveals the incentives that led him to write and the purposes that lay behind his treatise.

> Cum autem opera picturaeque meae a splendidissimo et fulgentissimo sidere, et maiore nostri temporis luminare optimi genitoris tui totum quicquid habent claritatis assumpserint: non ab re visum fuit opusculum quod in hoc ultimo aetatis meae calculo, ne ingenium inertia torpesceret, in mathematica de quinque corporibus regularibus edidi numini tuo dedicare, ut et ipsum ex obscuritate sua a claritate tua illustretur. Nec dedignabitur celsitudo tua ex hoc iam emerito, et fere vetustate consumpto agello, unde et illustrissimus genitor tuus, uberiores percepit, hos exiles et inanes fructus suscipere, et libellum ipsum inter innumera amplissimae tuae, paternaeque bibliothecae volumina penes aliud nostrum de Prospectiva opusculum, quod superioribus annis edidimus, pro pedissequo et aliorum servulo, vel in angulo collocare. Non enim solent non admitti quandoque in opulentissima et lautissima mensa, agrestia, et a rudi et inepto colono poma suscepta. Poterit namque, saltem sui novitate, non displicere. Etenim licet res apud Euclidem, et alios geometras nota sit, per ipsum tamen nuper ad arithmeticos translata est. Eritque pignus et monumentum mei in te, inclitamque prosapiam tuam antiqui amoris et perpetuae servitutis [3].

We learn here of Piero's advanced age and frail physical

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[2] *De corporibus regularibus,* p. 488. Cf. VITRUVIUS, *De architectura* (III, Preface).

[3] *De corporibus regularibus,* p. 488.

condition. And two other facts important for understanding the *Libellus* emerge in the dedication. First, Piero wished to have the *Libellus* shelved together with the *Prospectiva pingendi* in the Urbino Library, and, second, Piero believed his achievement in the *Libellus* to be the application of arithmetical principles to the geometrical theorems of Euclid.

Piero's wish that his two treatises stand together indicates that he intended the study of the *Libellus* to be associated with that of the *Prospectiva pingendi.* And, indeed, the works are united by a similarity of structure and content. The relationship of the measurement of regular and irregular bodies to the study of perspective was touched on in the preceding discussion of the *Trattato d'abaco.* However, it is the perspective treatises of Piero's successors, with their demonstrations in constructing and rendering in perspective stereometrical solids that make this connection explicit.

The application of arithmetical or algebraic exercises to Euclidean theorems, the novel aspect of Piero's treatise, is based on principles already worked out in his *Abaco.* But, while much of the *Libellus* derives from the *Abaco,* it represents a further development in Piero's mathematical studies, for a new consideration of Book XV of Euclid's *Elements* forms the basis of one section of the treatise.

1. *Regular bodies: The « Libellus de quinque corporibus », Parts I-III*

Over half of the exercises in the *Libellus,* Parts I-III, had previously been demonstrated in the geometry section of Piero's *Abaco* (fols. 80r-120r) [4]. In Part I of the *Libellus* — which treats two-dimensional figures, triangles, squares, pentagons, hexagons, octagons, and circles — some thirty-three of the

[4] *Trattato d'abaco,* ed. G. Arrighi (Pisa 1970).

fifty-five exercises can be found in the *Trattato d'abaco* (nos. 3, 4, 8, 11, 14-25, 28-30, 34-40, 42-47, 49). Fewer than half of these, around sixteen, have been reworked (nos. 3, 14, 20, 24, 25, 28, 29, 35-41, 43, 44). Seventeen of the problems have been taken over much as they were (nos. 4, 8, 11, 15-19, 21-23, 30, 34, 45-47, 49). For instance, *Libellus*, I, 15 reads:

> Eius quadrati cuius superficies est bis tanta quanta sunt latera sua quatuor, latus invenire?
> Habes in Algebra quod quadrata figura capitur pro censu, et suum latus habetur pro radice, id est pro re. Igitur dic sic. Est census qui est 4 res, quia aequat duplum 4, quod est 8 res. Et regula dicit quod partiaris res per census; et res tantum valet quantum inde provenit. Partiare 8 [res] per 1 [census] eveniet 8; et 8 valebit res quae posita fuit pro latere quadranguli: igitur fuit 8. Multiplica in se 8 fient 64. Et sua quatuor latera, quorum quodlibet, 8, efficit 32, et quadratum est 64, qui est duplum ad 32, qui sunt quatuor sua latera, quod propositum fuit [5].

This is clearly based on the *Trattato d'abaco,* folio 83r (2).

> Egl'è uno quadrato che la superficie sua è doi cotanto che i suoi 4 lati. Domando quanto è il suo lato.
> Tu ài nell'argebra che il quadrato se intende il censo e il suo lato se intende radici cioè la cosa; adunque dirai che meço censo sia equale a 4 cose. Dunqua 1/2 censo è equale a 4 [cose]; reduci ad 1 censo, arai 1 [censo] equale ad 8 cose; parti le cose per li censi ne vene 8, et 8 vale la cosa. Noi dicemmo che uno lato era 1 [cosa] adunqua fu 8; montiplica 8 in sé fa 64 et i suoi 4 lati, cioè 8 per lato, fa 32; adunqua il quadrato, che è 64, è doi cotanto che 32, che sono i 4 lati.

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[5] The *Libellus* problems are here numbered as they appear in Mancini's book, and in the Vatican 632 manuscript. I have numbered each of the exercises in Piero's *Abaco* in parentheses, according to the order of its appearance on the individual folio, following Arrighi's folio numeration. See Table II for the correspondences between the problems in the *Libellus* and the *Trattato d'abaco.*

Similarly, the foundation for the *Libellus,* I, 49, lies in the *Trattato d'abaco,* fol. 94r (1). Thus, in the *Libellus* we read:

> Si de diametro circuli, qui est ulnarum 10, linea ulnarum 9 1/2, resecant ulnas 3; in qua sui parte linea dividat quaerendum est. Sic age: Multiplica invicem diametri partes, quarum una est 3, altera 7. Multiplica per 7, erit 21. Nunc si dicas: fac de 9 1/2 partes duas huiusmodi, quarum ad invicem facta multiplicatio reddat 21. Ponamus partem unam esse 1 rem, et alteram 9 1/2, dempta 1 re. Multiplicata semel una res in 9 1/2, deducta 1 re, erunt res 9 1/2, dempto 1 censu. Et tu vis 21. Instaura partes: habebis 9 1/2 aequantem censum 1 et 21 numerum. Dimidiando, res erunt 4 3/4; multiplicatae in se, fient 22 9/16. Deduc numerum, qui est 21: remanet 1 9/16, cuius radix, deductis rebus dimidiatis, valet res, quae fuit de partibus lineae. Et altera fuit 4 3/4 addita radice ipsius 1 9/16. Itaque habes quod pars una est 4 3/4, remota radice ipsius 1 9/16. Et alia 4 3/4 addita radice ipsius 1 9/16, id est una 3 1/2, alia 6.

And, the corresponding passage in the *Abaco,* fol. 94r (1), is as follows:

> Egl'è uno tondo che il suo diametro è 10, et una linea che è 9 1/2 il sega, cioè ne tagla 3 bracci none ad angulo recto. Domando in che parte è devisa la linea dal diametro.
> Fa' così. Montiplica le parti del diametro l'una con l'altra, cioè 3 via 7 fa 21. Hora di' così: devidi 9 1/2 in tale parte che montiplicata l'una per l'altra facci 21. Mecti una parte essere 1 cosa e l'altra 9 1/2 meno 1 cosa, hora montiplica 1 [cosa] via 9 1/2 meno 1 [cosa] fa 9 1/2 [cosa] meno 1 censo, e tu voi 21. Restora le parti; da' ad omni parte 1 censo, arai 1 [censo] et 21 equale ad 9 1/2 [cose]. Demeçça le cose sirano 4 3/4, montiplicale fa 22 9/16, tranne il numero che è 21, resta 1 9/16. E la radici de 1 9/16 meno del dimeçamento de le cose, che fu 4 3/4, vale la cosa. Et fu dicto che una parte era 1 [cosa], dunqua fu 4 3/4 meno radici de 1 9/16, e l'altra parte fu 4 3/4 più radici de 1 9/16.

Aside from these correspondences, the Vatican *Libellus,* Part I, contains new references to Euclid in exercises 14, 34, 39, and 46, ones which do not appear in the corresponding problems in the *Trattato d'abaco.*

Part II of the *Libellus* teaches the measurement of the regular bodies (the tetrahedron, cube, octahedron, dodecahedron, and icosahedron) contained in a sphere. Of the thirty-seven exercises in Part II of the *Libellus,* twenty-six can be located in the *Trattato d'abaco* (nos. 4-16, 18, 21, 26-36). Fourteen of these have been reworked to some degree (nos. 4, 6, 11, 12, 14-16, 18, 21, 26-28, 30, 35). Twelve remain the same (nos. 5, 7-10, 13, 29, 31-34, 36). Several examples will illustrate these cases.

> Viginti basium corporis triangularium aequilaterarum cuius superficies est 200 brachiorum, quantum sit latus est perquirendum. In superiore dictum est quod si latus unius basis est brachiorum 4, cathetus est radix 12. Et eius basis superficies erit radix 48, sicut habes per IIam primi. Nunc habes quod 20 basium est 200 ulnarum: ideo partire 200 per 20, evenient 10, et 10 est superficies unius basis, quia proportio a superficie ad superficiem est dupla ad proportionem unius lateris unius superficiei, ad latus alterius superficiei quando sunt similia. Itaque dic: si radix 48 superficiei dat de latere 4, quantum dabit 10 de superficie? Reducas 4 ad radicem radicis, fiet 256. Et refer 10 ad radium fiet 100. Dic itaque: si 48 de superficie dat de latere 256, quantum dabit 100 de superficie? Multiplica centies 256, fient 25600: partire per 48, evenient 533 1/3. Et radix radicis erit per quodlibet latus corporis 20 basium, cuius superficies est 200 ulnarum.

This is *Libellus*, II, 34, and it is readily perceived that it is only a restatement of the *Trattato d'abaco,* fol. 113r (2), which follows.

> Egl'è uno 20-baxe triangulare equilatero, che la superficie sua è 200 bracci. Domando de' suoi lati.
> Tu ài che se il lato de una basa è 4 bracci, il catecto è radici

de 12, e la superficie de quella baxa è radici de 48. E tu ài che 20 braci è la superficie de' 20-baxe, però parti 200 per 20 ne vene 10 et 10 è superficie de una baxa. Et perché la proportione de la superficie a la superficie è doppia a la proportione del lato al lato de quelle superficie, però di': se radici de 48 de superficie dà de lato 4, che darà [10] de superficie? Reca 4 a radici de radici fa 256, et reca 10 a radici fa 100; montiplica 100 via 256 fa 25600, parti per 48 ne vene 533 1/3. Et la radici de la radici de 533 1/3 sirà per lato il 20-baxe triangulare, che la superficie sua è 200 bracci.

The following quotation illustrates a different situation.

Quando octobasis triangularis in quadratura esset 400, quaeritur de diametro sperae illum continentis.
Age sic. Invenias speram, cuius diameter notus sit, et hic sit 7, qui per XXIIIIam huius dat de quadratura octobasis 57 1/6. Reducas 7 ad radicem cubam, fiet 343. Dicas igitur sic. Si 57 1/6 quadraturae dat de diametro 343, quantum dabit 400 de quadratura? Multiplica 343 cum 400, evenient 137200, quem partiare per 57 1/6, evenient 2400. Et radix cuba huius 2400 est diameter sperae quae octobasim circumscribit, quadraturam habentem quantitatem 400.

This is *Libellus*, II, 26, which has been completely reworked from *Trattato d'abaco*, fol. 114v (1), which is given below, though the answer remains the same.

Egl'è uno 8-baxe triangulare equilatero, che è quadrato 400 bracci. Domando del diametro dela spera che il circumscrive. Tu ài, per la passata, che 1333 1/3 de quadratura dà de diametro 20; reca 20 a radici cuba fa 8000, però di': se 1333 1/3 dà 8000, che darà 400? Montiplica 400 via 8000 fa 3200000, il quale parti per 1333 1/3; fa' terçi de le parti, arai 9600000 [a] partire per 4000 che ne vene 2400. Et la radici cuba de 2400 sirà il diametro de la spera che contene l'octo-baxe triangulare, che la sua quadratura è 400 bracci; ch'è il proposto.

In contrast, the first part of Part III (exercises 1 through 13) reveals Piero breaking new ground, and it evidences his total immersion in Book XV of Euclid's *Elements* [6]. Problems 14 through 29 comprise the exercises on the sphere to which Piero refers in his introduction. He prefaces this section with a paragraph in which he defines the sphere according to Euclid and Theodosius — a sphere is a round body, its greatest circumference is determined by its axis, and its area is determined by its axis and circumference [7].

Nonetheless, of the sixteen problems Piero enumerates on the sphere (Part III, nos. 14-29), thirteen are already present in the *Trattato d'abaco* (nos. 15-17, 19-25, 27-29). Five of these problems are much the same in both treatises (nos. 17, 23-25, 28). We may compare, for instance, *Libellus,* III, 24, with the *Trattato d'abaco,* fol. 118v (1). No. 24 of the *Libellus* reads:

> Speram cuius axis est 14, linea plana 9 ulnarum dividat. Quo in loco resecet axem pervidendum est.
> Habes speram ABCD, cuius AD est axis, et linea BC scindit axem in puncto E. Et quia resecat ad angulum rectum divisa est linea BC per aequa in puncto E. Igitur BE est 4 1/2, quae est medietas BC, quae est 9. Multiplicato 4 1/2, in se, efficit 20 1/4. Nunc si dicas de diametro sive axe, quae est 14, facias partes duas, quarum ad invicem multiplicando reddat 20 1/4. Ideo dic quod una pars sit 1 res, altera sit 14, dempta 1 re. Multiplices 1 rem cum 14, dempta 1 re, fient 14 res, remoto 1 censu: et tu vis 20 1/4. Restaura partes: habebis 1 censum, et 20 1/4 numerum aequantem 14 res. Dimidiando res, erunt 7, quae in se multiplicatae reddunt 49, de quo deme numerum, qui est 20 1/4, supersunt 28 3/4. Et radicem 28 3/4, demptis rebus di-

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[6] *Euclidis Megarensis philosophi acutissimi ... Opera,* ed. L. Pacioli (Venice 1509), fols. 141r-144v.

[7] *De corporibus regularibus,* p. 549: « *Est igitur spera, auctore Euclide, transitus circumferentiae dimidii circuli, quotiens fixa diametro, quoasque ad locum suum redeat circumducitur. A Theodosio autem sic deffinitur: Spera est solidum corpus una superficie contentum, in cuius medio punctus est a quo omnes lineae ductae ad circumferentiam sunt aequales.* »

midiatis, valet res. Ergo pars una fuit 7, dempta radice 28 3/4. Altera pars 7, addita radice 28 3/4. Scindit igitur de axe 7, deducta radice 28 3/4.

It is a simple matter to see in this problem the reflection of the corresponding problem in the *Trattato d'abaco,* fol. 118v (1).

Egl'è una spera che il suo diametro è 14 bracci;meno una linea piana, ch'è 9 bracci, segante l'assis de la spera ad angulo recto. Domando quanto taglarà de l'assis.
Tu ài la spera ABCD et AD è diametro; et BC il sega in puncto E et, perché la sega ad angulo recto, è devisa BC per equali in puncto E; dunqua BE è 4 1/2, ch'è una parte de 9; che montiplicato 4 1/2, ch'è una parte de 9; che montiplicato 4 1/2 in sé fa 20 1/4. Hora di' mo così: famme del diametro, ch'è 14, do' parti che montiplicata l'una per l'altra faccia 20 1/4. Dunqua mecti una parte 1 cosa e l'altra 14 meno 1 cosa; montiplica 1 cosa via 14 meno 1 cosa fa 14 cose meno 1 censo, et tu voi 20 1/4. Restora le parti; da' ad omni parte 1 censo, arai 14 cose equale ad 1 censo e 20 1/4 numero. Demeçça le cose sirano 7, montiplicale in sé fa 49, tranne il numero ch'è 20 1/4, resta 28 3/4. Et la radici de 28 3/4 meno del dimeçamento de le cose, che fu 7, vale la cosa. Dunqua una parte fu 7 meno radici de 28 3/4, l'altra fu 7 più radici de 28 3/4.

Eight of the sixteen problems present in both texts have been reworked (nos. 15, 16, 19-22, 27, 29). *Libellus,* III, 19, for example, reads:

Si de quadratura sperae, cuius axis est 7, fiat quadratura cubi, quantum sit eius cubi latus perquirendum est.
Reducas speram in quadrum, cuius axis est 7. Et per XVIam huius habes quod eiusmodi sperae quadratum est 179 2/3. Igitur erit cubi latus radix cuba 179 2/3. Alia ratione potest demonstrari, videlicet cum proportione. Ea enim proportio est inter latus cubi et diametrum sperae eiusdem quadraturae, quae est inter radicem cubam 343, et radicem cubam 179 2/3. Nam

> si reducas 7 ad radicem cubam, quae est axis sperae, fient 343. Et scis quod cubus, id est quadratura eius, est tamquam 21 ad 11 ad sperae quadraturam. Ideo multiplices 343 per 11, fient 3773: partire per 21, evenient 179 2/3. Itaque radix cuba 179 2/3 est latus cubi quod quaerebamus.

The *Trattato d'abaco*, (fol. 117v. [1]), on the other hand, presents the following methods of solution.

> Egl'è una spera che il suo diametro è 7 bracci; voglo de la sua quadratura fare uno cubo. Domando de' suoi lati.
> Quadra la spera che il suo diametro è 7, reca a radici cuba fa 343; et, perché la spera è 11/21 del suo cubo et il suo cubo è 343, perhò montiplica 11 via 343 fa 3773, il quale parti per 21, ne vene 179 2/3; però sirà il cubo radici cuba de 179 2/3. Possiamo fare per altra via, dire così: sì conmo ài di sopra che il diametro de la spera ch'è radici cuba de 122 2/11 dà il lato del cubo radici cuba de 64, dunqua che te darà il diametro ch'è 7? Reca 7 a radici cuba fa 343, montiplica 64 via 343 fa 21952, il quale parti per 122 2/11 ne vene 179 2/3, e la radici cuba de 179 2/3 sirà il cubo per lato; sì conmo di sopra.

In addition to all this, further references to Euclid are found in the *Libellus*, Part III (nos. 23, 25, 27, 28), and there others to Archimedes (nos. 15, 16, 20, 23).

The above citations from the two treatises show that Piero does not reproduce problems already worked out in the *Abaco* in the *Libellus* without redefining their methods of solution and improving their accuracy. One cannot fail to notice that the *Libellus* is frequently more concise than the *Trattato d'abaco* and that it frequently adds passages to clarify mathematical procedures. At times, apparently dissatisfied with the form of solution utilized in the *Abaco*, Piero changes it in the *Libellus*. Frequently the resulting numerical answer differs. Sometimes, though the method of solution changes, Piero arrives at the same answer (*Libellus*, II, 30 and *Tratta-*

to d'abaco, fol. 111r [1]). In other cases, where two methods of solution were given in the *Abaco* (for example, fol. 90r [1]), Piero chooses to reproduce only the first in the *Libellus* (I, 28). But more often than not the problems in the *Trattato d'abaco* are executed with relatively little change in the *Libellus.*

The consistent use of shorter sentences, more concise phrases, and more exact punctuation makes the whole of the *Libellus*, in contrast to the *Abaco*, appear clearer, more crystallized, and more ably constructed. Much of this impression must, however, be attributed to the translator of Piero's manuscript, Matteo da Borgo, evidently a capable Latinist, for most likely the Italian manuscript he worked from read much like the *Trattato d'abaco* [8].

It would be a mistake, however, to conclude that Piero assembled the *Libellus* mainly from the earlier *Trattato d'abaco*. Part III, entitled « Incipit tertia pars quinque corporum. De dimensione laterum ipsorum, divisionibus axis, superficiebus, transmutationibusque unius corporis in aliud, etc., » represents entirely new work in exercises 1 through 13, and it reveals a new and penetrating study of Book XV of Euclid's *Elements*. This section, Piero's « *tertia pars* », is introduced with a statement, which announces Piero's intention to ex-

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[8] LUCA PACIOLI, in his *Summa de arithmetica, geometria, proportioni et proportionalità* (Venice 1494), fol. 68v, reports that « *maestro Matteo* » translated the *Prospectiva pingendi,* and it is very likely that he also translated the *Libellus*: « *El sublime pictore ... maestro Piero de li franceschi ... hane in questi di composto un degno libro de ditta prospectiva ... El qual lui feci vulgare: e poi el famoso oratore: poeta: e rhetorico: greco e latino (suo assiduo consotio: e similmente conterraneo) maestro Matteo lo recco a lengua latina ornatissimamente de verbo ad verbum: con exquisiti vocabuli.* » For Matteo Cioni da Borgo, see L. COLESCHI, *Storia della città di Sansepolcro* (Città di Castello 1886), p. 227; P. FARULLI, *Annali e memorie dell'antica e nobile città di S. Sepolcro* (Foligno 1713), p. 64. See below, Appendix II, for an analysis of the respective Latin and Italian texts of the *Libellus* and *Abaco.*

plain how the sides of various bodies are related to one another and how the one may be contained within the other, and to treat the area of the sphere and its square, and still other questions.

> Nunc in hac tertia parte volumus, sicut in principio dixi, exponere quanta sint latera corporum unius ab altero contentorum, et quot contineantur in uno, et quot in altero. Deinde dicam de superficie sperae atque quadratura, et de quibusdam axis divisionibus et superficiebus, et quadraturis a planis lineis factis, et transmutationibus sperae in cubos, et cuborum in speram, et pariter sperarum in conos et conorum in speras [9].

The latter promise refers to exercises 14 through 29 of Tract III, on the sphere. The first part, describing the measurement of one regular body located in another, is found in the first thirteen exercises. In the *Trattato d'abaco* there are no exercises that measure one of the solids placed within another, nor are there references to Book XV of the *Elements*. Thus the first thirteen exercises of Part III of the *Libellus* represent a further stage in Piero's mathematical development.

This work results from a consideration and elaboration of Book XV of the *Elements,* and this may be seen by examining two of Piero's exercises. No. 2 reads, « If a tetrahedron is placed in a cube whose side equals 12 braccia, what is the side of the tetrahedron? » Piero solves the problem by first drawing diagonals through one face of the cube. « We know from Euclid, I, 47, [i.e., the Pythagorean theorem] », he writes, « that if the sides of a square equal 12, its diagonal will equal the square root of the squares of two sides added together », that is, $\sqrt{288}$. « And $\sqrt{288}$ will, furthermore, be the side of an inscribed tetrahedron according to Euclid, XV, 1 ». The first

[9] *De corporibus regularibus,* p. 540.

proposition of Euclid, XV, 1, shows the relationship of a tetrahedron inscribed in a cube: « *Intra propositum cubum corpus habens quatuor bases triangulas equalium laterum designare* ». In this proposition a tetrahedron is inscribed in a cube so that each side of the tetrahedron equals the diagonal of each face of the cube. Hence knowing from Euclid, XV, 1, the relationship of the side of a cube to the side of a tetrahedron inscribed in it, Piero applied an arithmetical procedure.

A second problem from Piero's *Libellus,* no. 3, reads, «If an octahedron is contained by a cube whose side is 12 braccia, what is the side of the octahedron? » To find its solution Piero first places a tetrahedron, as in problem 2, in a cube, and its side is shown (as above) to be $\sqrt{288}$. « Divide $\sqrt{288}$ », he writes, « into four equal parts and you have $\sqrt{72}$ ». And, « $\sqrt{72}$ is the side of an octahedron contained in a cube whose side is 12 braccia ». This proposition, too, is demonstrated in Euclid, XV, 3: « *Intra cubum assignatum figuram octo basium triangularium equalium laterum constituere cubo intendimus inscribere octocedron* »[10]. In proposition 3 the relationship between the side of a tetrahedron and that of an octahedron, i.e., the side of an octahedron is one fourth that of a tetrahedron, is explained. We read in Euclid.

> Constat enim ex ratiocinatione prima latera cuncta ipsius inscripte pyramidis esse diagonos basium cubi. Et ex ratiocinatione premisse liquet cunctos angulos octocedri in hac piramide distincti esse in lateribus ipsius piramidis. Quare manifestum est omnia angularia puncta huius octocedri esse in basibus assignati cubi. Igitur ex diffinitione habemus propositum.

What Piero has, in effect, done in the first part of Part III is to apply arithmetical problems to the fixed geometrical pro-

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[10] *Opera,* ed. Pacioli, fols. 141r-142r.

portions that exist among the regular bodies inscribed in one another as defined in Book XV of the *Elements*. In fact, all of Piero's problems, save one, relate directly to Euclid's text. The following table outlines these correspondences exhaustively [11].

Libellus, III, no.:	*Elements*, XV, proposition:
1. Octahedron in a tetrahedron	2
2. Tetrahedron in a cube	1
3. Octahedron in a cube	3
4. Icosahedron in a cube	—
5. Cube in an octahedron	4
6. Tetrahedron in an octahedron	5
7. Cube in a dodecahedron	8
8. Tetrahedron in a dodecahedron	10
9. Octahedron in a dodecahedron	9
10. Icosahedron in a dodecahedron	7
11. Cube in an icosahedron	11
12. Tetrahedron in an icosahedron	12
13. Dodecahedron in an icosahedron	6

2. *Irregular bodies: The « Libellus de quinque corporibus », Part IV*

The exercises in Part IV of the *Libellus*, on irregular bodies, are mostly abstract and mathematical in nature, but a few can be related to the study of architecture. And, in their entirety, the problems relate to the study of perspective. The

[11] In the *Divina proportione*, Part I, Chapters 35-45, PACIOLI demonstrates the proportional relationship of one regular body inscribed in another. With the exception of Book XV, prop. 2, an octahedron in a tetrahedron, his eleven examples follow Euclid's, even in their ordering.

essential components of problems 2-9, 12-15, and 18, are found in the *Trattato d'abaco*. These exercises mainly measure irregular geometrical solids formed from triangles, squares, hexagons, pentagons, and octagons, all placed in a sphere. For example, if in a sphere whose axis is 12 braccia, Piero writes in exercise 6, an irregular body composed of four triangles and four hexagons is placed, what are the sides, surfaces and square of this solid? The *Trattato d'abaco*, fol. 107r-v, presents a similar problem. If, in a sphere whose axis is 6, an irregular body of four triangles and four hexagons be placed, what are its sides? And then, asks Piero, what is its square?

The first problem in Part IV of Piero's *Libellus* is, however, one with which architects often dealt — the measurement of a 72-faced solid, a solid made up of twenty-four triangular and forty-eight four-sided polygonal faces, having unequal sides and angles. Piero writes that Johannes Campanus, the medieval translator and commentator of Euclid, taught how to construct this body in Book XII, proposition 14, of the *Elements*. Though Piero does not mention its usefulness in architectural constructions, Luca Pacioli testifies to its importance in this context in the *Divina proportione,* Part I, Chapter LIV, entitled « Del corpo de 72 basi piano, solido, e vacuo » [12]. Half of this body, Pacioli writes, the half that resembles a hemisphere, is found in the « *antico templo pantheon* ». A quarter of the solid is used in the architecture of a tribune, or apse, as in Milan, in the chapel of S. Satiro and in S. Maria delle Grazie: « *E questo 72 basi molto dali architetti sia frequentato in loro dispositioni de hedifitii per essere forma asai acomodata maxime dove occurrese fare tribune o altre volte o vogliamo dire cieli* » [13]. Pacioli's remarks on this topic relate to the discussion of architectural representations in Pie-

[12] *Divina proportione* (Venice 1509), fol. 16r.
[13] Ibid.

ro's writings, for in the *Prospectiva pingendi* Piero furnishes directions for rendering, in perspective, « *una cupola per ragione, la quale fusse commo uno quarto de una palla dal canto concavo e fusse devisa in quadrati nelli quali fussero rossoni* » [14].

Among the architectural elements which Piero showed how to foreshorten in the *Prospectiva pingendi,* there were also an eight-sided and a sixteen-sided column and a cross vault — « *una volta in crociera* » [15] (fig. 4). Similarly, Piero demonstrates in Part IV of the *Libellus* the correct mathematical measurement of a round column and a vault « *per modum crucis* » [16] (fig. 5). These three exercises relate to the architect's concern with architectural description, as well as to the painter's concern with pictorial perspective. The ability to measure vaults, columns, and other architectural parts entails knowing « *la forza de le linee et degli angoli* », necessary for perspective studies [17].

Concerning perspective, the loquacious Luca Pacioli is more informative about the connection between regular and irregular bodies and perspective science than Piero is. When he translated and appended Piero's *Libellus de quinque corporibus regularibus* to his *Divina proportione* of 1509, Pacioli inserted the following lines between exercises 4 and 5 of Part IV of the *Libellus,* apologizing to the reader that the solids do not always appear in the margin of the treatise because they need to be drawn by one well practiced in the science of

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[14] *De prospectiva pingendi,* ed. G. Nicco Fasola (Florence 1942), I, 202.

[15] Ibid., pp. 104, 112, 122. There are further architectural demonstrations in the *Prospectiva pingendi* regarding the base of a column (p. 148); the capital of a column (p. 157); a six-sided well (p. 106), and « *uno tempio de octo facce* » (p. 118).

[16] *De corporibus regularibus,* p. 571: « *Est quaedam testudo, seu volta, per modum crucis, quae est pro qualibet facie 8 brachia, et in altitudine 4 brachia, tam in summitate arcuum quam in medio voltae: quaeritur de superficie concava?* »

[17] *Prospectiva pingendi,* p. 128.

perspective. « *Lectore non te maraviliare se de simili corpi composti de diverse e varie base non te se mette sempre in margine loro figure conciosia ch' le sieno difficilime farle in desegno pero che bisogna che sieno facte per mano de bono prospectivo quali non si posano sempre havere...* »[18].

For the most part the exercises presented in Part IV of the *Libellus* grew out of the subjects already treated in the *Trattato d'abaco* and the *Prospectiva pingendi,* and the relationship of the three treatises to one another is again closer than a superficial reading suggests. The same themes, such as the golden section and the regular bodies, are treated both in the *Abaco* and the *Libellus,* and the application of the measurement of regular polygons and polyhedra to painting is demonstrated in the *Prospectiva pingendi.* The exercises in all three treatises derive from Piero's unifying interest in correctly measuring and correctly representing objects found in nature. Piero's source, Euclid's *Elements,* remains moreover his life-long stimulus.

The rules and methods for measuring two-dimensional polygons in Part I of the *Libellus* and three-dimensional polyhedra in Part II, as well as the definition of the relationship of one regular body to another and the exercises on the sphere in Part III, lay the groundwork for the more complex irregular bodies discussed in Part IV. Many of these irregular solids relate to the familiar objects in nature that fall to the painter to represent. Piero apparently considered the study of the progressively complex solids, such as those in the *Abaco* and *Libellus,* a prerequisite to the accurate representation of the more complex forms of nature. In this context, it is worth recalling the example of Alberti's *Elementa picturae,* a mathematical treatise reworked for artists, which contains instructions in drawing regular and irregular polygons, which in

[18] *Divina proportione,* fol. 22r.

turn constitute the rudiments of perspective. Thus it seems reasonable to see the same connection that Piero saw between his *Libellus de quinque corporibus regularibus* and his *De prospectiva pingendi* when he wrote to Guidobaldo hoping that his two works would be preserved side by side in some corner of the copious Montefeltro library, there to bask in the reflected splendor of the young duke.

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Many of the figures presented in Part IV of the *Libellus* concern the measurement of building parts — columns, vaults, apses, domes, and the like. The correct representation of these was required of the architect in presenting his plans, of the painter in defining the architectural ambients for his histories, and of the scenographer in designing and erecting his stage settings. Furthermore, it was of considerable consequence for the punctilious designs of intarsia makers.

Luca Pacioli's *Summa arithmetica* (which contained Piero's geometrical exercises from the *Abaco*) was written for practitioners of the mechanical arts, among others, intarsia craftsmen, who needed to be well-versed in the rules of proportion that governed their compass and rule [19]. Later, in his *Divina proportione,* Pacioli discloses Piero's connections with intarsia workers, writing that the famous Lorenzo da Lendinara was « *caro quanto fratello* » to Piero [20].

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[19] *Summa arithmetica,* Dedication to Guidobaldo: « *De larti tutte Mecaniche. discorrendo in tutti exerticii. e mistieri. non si vede oculata fide. Che toltoli de mano la squadra. El sexto. con la lor proportione non sanno che si peschino. Chi de tarsia. si nobilmente con tanta diversita de legnami per tutto apieno lunico. U.D. palaçço ha disposto. se non le doi linee curva e retta: con suoi pontuali termini proportionata.* »

[20] *Divina proportione,* fol. 23r: « *e anco con quella prometto darve piena notitia de prospectiua medianti li documenti del nostro conterraneo e contemporale di tal faculta ali tempi nostri monarcha maestro Petro de franceschi dela qual gia feci dignissimo compendio e per noi ben apreso. E del suo caro quanto fratello Maestro Lorenço canoço da Lendenara: qual*

The principal motifs of early intarsia panels were usually simple objects of a geometrical character, or views of piazzas and buildings, rendered in perspective. In fact, the virtuoso perspective techniques of the woodworkers of the Quattrocento resulted in the appellation « *maestri di prospettiva* », which was by and large the term applied to intarsia workers [21]. The substantial market they had to satisfy is reflected in the fact that there were in Florence alone eighty-four *botteghe* for intarsia and woodcarving in 1470 [22]. Moreover, fifteenth-century paintings, as well as household inventories, reveal that intarsia was of first importance for domestic furnishings. And in the ecclesiastical sector the situation was the same. The significant number of intarsia decorated choir stalls that have survived the ravages of time testify to their popularity. The simple designs, or patterns, for these works, handed from the master (often an architect or painter) to the crafsman, have not survived, but we can reconstruct some aspects of the appearance of such cartoons from the illustrations to the *Prospectiva pingendi* and the *Libellus de quinque corporibus regularibus.*

For example, the *mazzocchio* in the *Prospectiva pingendi* (fig. 6) compares well with those found in the Lendinara choir stall in Modena (fig. 7); similarly, the hexagonal well on a base

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medesimamente in dicta faculta fo ali tempi suoi supremo chel dimostrano per tutto le sue famose opere di intarsia nel degno coro del Sancto a Padua ....»

[21] G. C. Romby, *Descrizioni e rappresentazioni della città di Firenze nel XV secolo, con la trascrizione inedita dei manoscritti di Benedetto Dei* (Florence 1976), p. 73. In the « Memorie » of Benedetto Dei (Cod. Riccardiano 1853), fol. 90r, the *intarsiatori* of Florence are listed « *Maestri di Prospettiva in Firenze tutti Fiorentini l'anno 1470.* » He names fourteen adding, « *et altri maestri che sono n° 19.* »

[22] Ibid., p. 49, from Benedetto Dei's « Cronaca fiorentina, dal 9 dicembre 1430 al 1480, » (Florence, Archivio di Stato, Manoscritti, no. 119), fol. 32v: « *Florentia bella à 66 botteghe di speziali e à 84 botteghe di legnaiuoli di tarssie e 'ntagliatori.* »

illustrated in the *Prospectiva pingendi* (fig. 8) seems almost to be the model for the six-sided fonts on the seats of the Modena stalls (fig. 9). The octagonal baptistery in Modena (fig. 11), too, grows out of the «*tempio de octo facce*» shown foreshortened in the *Prospectiva pingendi* (fig. 10). As a last example, one may compare the illustration of the 72-faced solid in the *Libellus* (fig. 12) and the *cupola* in the *Prospectiva pingendi* (fig. 13), with the intarsia view of a choir in Monte Oliveto by Giovanni da Verona (fig. 14). The same master translated the regular and irregular solids invented by Piero (and published by Pacioli, figs. 20, 21), into skillful wood intarsia panels in S. Maria dell'Organo in Verona (fig. 22).

This illusionary world of scientific tools, musical instruments, sacred and secular still-life objects, set beside street perspectives and in conjunction with purely stereometrical bodies, represents the triumph of a conception of space based on numerical and geometrical rules. These fifteenth-century intarsia panels are, in a way, visual realizations of the mathematical exercises in Piero's treatises. Both in what they represent and in their methods of execution they are an emblem of the union of art and science in the Renaissance.

IV.

Applications of the Art of Measurement

Piero's stereometric conceptions and inventions were first made widely available to artists and treatise writers through Luca Pacioli's publication of them. In 1494 Pacioli appended the stereometric exercises in the *Trattato d'abaco* to his *Summa de arithmetica, geometria, proportioni et proportionalità,* and in 1509 he published the *Libellus de quinque corporibus regularibus* in Italian at the end of his *Divina proportione.* In neither case did he acknowledge Piero's authorship [1]. Since most of Piero's manuscript writings eventually disappeared from view, Pacioli's « plagiarism » was essential to the widespread dissemination of Piero's ideas, and it sowed the knowledge of geometric solids that became instrumental in the development of perspectival and proportional exercises in Italy and in the North. Indeed the theme of the regular bodies became a standard feature of later Renaissance treatises.

Although fundamental, the relationship of the abstract planar figures and the three-dimensional solids of Piero's mathematical treatises to his more famous perspective book is not as immediately apparent today as it was in the fifteenth and sixteenth centuries. Yet, as we saw, the taciturn Piero himself

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[1] See below, Appendix I.

implicitly related the study of regular bodies to the study of perspective when he requested that the *Prospectiva pingendi* and the *Libellus de quinque corporibus regularibus* be shelved together in the Urbino Library.

It was Luca Pacioli who made explicit the intrinsic connection that exists between the regular and irregular solids, on the one hand, and perspective, on the other. In his *Summa arithmetica* Pacioli lists artists with whom he had discussed the science of perspective — « *Quali sempre con libella e circino lor opere proportionando a perfection mirabile conducano* »[2] And he recommends Piero's *Prospectiva pingendi,* a « *dignissimo compendio, e per noi ben apreso* », to readers of both his own printed works[3].

Then in his *Divina proportione* Pacioli included fifty-nine full-page woodcuts of regular and irregular bodies drawn in perspective and based on models prepared by Leonardo da Vinci[4]. These geometrical polyhedra were rendered in solid, hollow, and stellated variations and accompanied by their proper Greek and Latin names. Immediately following these illustrations, Pacioli translated and appended Piero's *Libellus de quinque corporibus regularibus,* without identifying its author. To the translation, however, Pacioli added a paragraph of his own apologizing for not including in the margins illustrations of the geometrical solids he discussed, but these, he wrote, must be drawn « *per mano de bono prospettivo* »[5], and therefore he referred his readers back to the preceding designs of Leonardo. In so doing, Pacioli not only joined the study of perspective to his own work on regular bodies, but

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[2] *Summa de arithmetica, geometria, proportioni et proportionalità* (Venice 1494), Dedication.

[3] *Summa arithmetica,* fol. 68v; *Divina proportione* (Venice 1509), fol. 23r.

[4] See above, Chapter I, n. 14.

[5] *Divina proportione,* fol. 22r.

he also connected the geometrical exercises contained in Piero's *Libellus* to lessons in perspective.

Vasari, of course, had perceived an immediate connection between the mathematics of Piero and his painting, joined as they were through the science of perspective. For Vasari, Piero was a past master of regular bodies as well as of arithmetic and geometry, and his accomplishments in these sciences are illustrated by all his works, which are full of perspectives. But one work in particular shows this better than any other, a lost painting, or perhaps better, a drawing of a « *vaso* » shown in perspective. From Vasari's account it is clear that Piero's drawing utilized methods of constructing and foreshortening perfect curves on the basis of regular bodies, that is, methods founded in his mathematical studies. This is what Vasari says about the work: « *...un vaso in modo tirato a quadri e facce, che si vede dinanzi, di dietro e dagli lati, il fondo e la bocca: il che è certo cosa stupenda, avendo in quello sottilmente tirato ogni minuzia, e fatto scortare il girare di tutti quei circoli con molta grazia* » [6].

Understandably, this striking description has been associated with a drawing in the Uffizi of a faceted chalice composed of circles, polygons, and *mazzocchi* set in perspective, a drawing usually attributed to Paolo Uccello [7]. It recalls, too, Vasari's description of some of Uccello's favorite perspective exercises — « *mazzocchi a punte a quadri tirati in prospettiva per diverse vedute, e palle a settantadue facce a punte di diamante, e in ogni faccia brucioli avvolti su per e bastoni* » [8] — nothing less than solid, hollow, and stellated polyhedra drawn in perspective, exercises that entailed the same kinds of procedures as Piero's vase.

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[6] VASARI-MILANESI, II, 491.

[7] J. POPE-HENNESSY, *Paolo Uccello, Complete Edition,* 2nd ed. (London 1969), p. 27, plate 86.

[8] VASARI-MILANESI, II, 205.

The rapid dissemination of Piero's ideas and inventions was, to an extent, ensured by the public for whom Pacioli wrote his books. In the *Summa arithmetica* Pacioli addressed students of architecture and perspective, as well as designers of *tarsie,* and in his *Divina proportione,* « *ciascun studioso di Philosophia, Prospectiva, Pictura, Sculptura, Architectura, Musica e altre Mathematice* » [9]. Moreover, Piero's inadvertent publicist took pride in his friendships with artists. Among the many artists he engaged in long discussions over the science of perspective and proportion, Pacioli counted, in Venice, Gentile and Giovanni Bellini « *carnal fratelli* », and, in addition, a certain Hieronimo Malatini *prospectivo,* and elsewhere, Andrea Mantegna, Melozzo da Forlì, and Marco Palmezzano, and of course Piero, all names synonymous with perspective [10]. No doubt these artists and others sought out Pacioli for his advice on the scientific questions of measurement and proportion taught in his treatises. And, Pacioli had earlier taught arithmetic and geometry to the *scalpellini* and architects of Sansepolcro, for whom he composed a Vitruvian architectural text [11].

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Certainly one further artist whom Pacioli knew well was the painter of the impressive portrait of Pacioli and of the world that occupied his mind, a painting long in Urbino and now in Naples, which bears the inscription, « IACO. BAR. VIGENNIS P. 1495 », identifying its artist and its date (fig. 15). At a glance, the work seems almost a manifesto promoting an ongoing dialogue between art and mathematics. And, examined closely in its details, it reveals itself to be an accurate

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[9] *Divina proportione,* title page.

[10] *Summa arithmetica,* Dedication.

[11] See above Chapter I, n. 12. For PACIOLI, see H. STAIGMÜLLER, « Lucas Paciuolo, eine biographische Skizze, » *Zeitschrift für Mathematik und Physik,* XXXIV (1889), 81-102, 121-28.

illustration of Pacioli's artistic-mathematic colloquies, an illustration that indeed preserves, if not the substance of past conversations, at least the echoes of their essential themes.

In his portrait, now in Naples, Pacioli stands, as if in an *aula* before his artistic acquaintances and all the rest who would assemble to learn. Though dressed in his Franciscan habit, it is Pacioli the learned teacher of mathematics who confronts us. His table is laid out with all the trappings of his profession, the instruments and accessories of his various rôles — teacher, mathematician, translator, and scientific publicist. With one hand Pacioli indicates a passage in the book opened before him, and with the other, he illustrates its words with a geometric diagram drawn on a slate, inscribed « EVCLIDES » along the side of its frame. Before him, a heavy, magnificently bound tome bears the letters, « LI. R. LVC. BVR. », which identify it as work by Pacioli — « *Liber Reverendi Lucae Burgensis* ». On top of this book stands a dodecahedron made of wood, and to Pacioli's side a transparent crystal icosahexahedron is suspended. Across the table are scattered his chalk and sponge, his compass and protractor, and his pen, ink, and case, as it were, the tools of his trade.

Old inventories, the first in 1582, identify the elegant young man behind Pacioli as the Montefeltro prince, Guidobaldo [12]. This identification certainly results from an inscription that once ran along the bottom of the panel — « *Divo principi Guido* » — which a busy inventory compiler apparently took to mean that Guidobaldo was represented in the painting [13]. « *Divo principi Guido* » is, of course, a dative, and

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[12] See F. SANGIORGI, ed., *Documenti urbinati: inventari del Palazzo Ducale, 1582-1631* (Urbino 1976), pp. 40, 57, inventories of 1582 and 1599.

[13] Ibid., p. 205, inventory of 1631: « Un ritratto di un frate di San Bernardino con un giovane apresso vestito di peliccia all'antica: segnato al basso Divo principi Guido, in tavola. »

it records the dedication of the painting to the young prince. In much the same manner Pacioli had the year previous dedicated the *Summa* to him. Indeed the portrait bears no resemblance to authentic likenesses of the prince. Given the turn of the head, the side-long glance, and the age of the painter of the work (« *vigennis* »), it is tempting to believe that we see the artist's own self-portrait [14].

If we examine his picture more closely, we will see that the artist's mirror reflects not just the stage properties of Pacioli's classroom, but his lesson as well. On the opened pages in the bound manuscript that Pacioli takes as his text, only the words « LIBER XIII » can be made out. « LIBER XIII » can only refer to the last book of Euclid's *Elements,* the *locus classicus* of the regular bodies. The diagrams in Pacioli's book are legible, and, by comparing them with Pacioli's later edition of Euclid, we see that Pacioli is consulting his manuscript of Euclid's *Elements.* Indeed, he points to Proposition 8 of Book XIII, « *Omnis trianguli equilateri quod a latere suo quadra-*

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[14] Althought not visible in the Alinari photograph (34011), the young man in the Naples painting wears a rounded cap (see B. MOLAJOLI, *Il Museo di Capodimonte* [Naples 1961], pl. XVII) of a common type, which is also encountered among Vasari's portraits of artists. For instances of similar headgear, see W. PRINZ, *Vasaris Sammlung von Künstlerbildnissen* (Florence 1966), p. 107 (Carpaccio), and Nos. 23 (Lippo), 40 (Giuliano da Majano), 41 (Piero), 44 (Lazzaro Vasari), 59 (Ercole de' Roberti), 60 (Giovanni Bellini), 69 (Verrocchio), and 71 (Filippino Lippi). The Naples youth's fine costume is reproduced in nearly every detail in an apparently contemporaneous Venetian portrait of a *Youth with Fur* in Venice (Museo Correr), variously attributed to Gentile Bellini, Giovanni Bellini, Carpaccio, and Mocetto (G. Perocco, *Tutta la pittura del Carpaccio* [Milan 1963], n. 84 and pl. 192), indicating possibly a Venetian localization of the costume. The extravagant hairstyle of the Naples portrait may be again compared with Vasari's artist portraits; note particularly a self-portrait of Ercole de' Roberti reproduced in Prinz, p. 51. Three authentic portraits of Guidobaldo are in Vienna, Kunsthistorisches Museum, Ambraser Coll.; Florence, Uffizi; and in B. CASTIGLIONE, *Il Cortegiano,* Rome, Vatican, MS.

tum describitur triplum est quadrato dimidii diametri circuli a quo triangulus ipse circumscribit »[15], which states, « If an equilateral triangle be inscribed in a circle, the square on the side of the triangle is triple the square on the radius of the circle »[16]. This theorem is fundamental to the inscription of regular bodies in a sphere, for it establishes the proportional relationship of the equilateral triangle to a circle, a correspondence required for the eventual construction and inscription of the tetrahedron, octahedron, and icosahedron. In modern editions of the *Elements* this proposition is numbered 12 and immediately precedes the proposition concerning the construction of the simplest regular stereometrical figure, the tetrahedron, and its proportional relationship to the sphere. Looking then to Pacioli's slate, shown in perspective, we see that he has copied from the margin of his Euclid the diagram of this theorem (Proposition 8), a triangle inscribed in a circle, and he demonstrates the proof of the theorem to his listeners, who, we understand, are learning the inscription of regular bodies in a sphere[17] — a central theme from Piero's *Abaco* that was re-

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[15] *Euclidis Megarensis philosophi acutissimi ... Opera,* ed. L. Pacioli (Venice 1509), fol. 124r-v.

[16] *The Thirteen Books of Euclid's Elements,* ed. T. L. Heath, 2nd ed. rev. (New York 1956), III, 466-67.

[17] The line that Pacioli has just constructed and now indicates with his pointer, when continued to the circumference, will be equivalent to the radius of the circle into which the equilateral triangle is inscribed, and it is upon this element that the proof of the theorem depends. To demonstrate the proof, as may be seen on Pacioli's slate, the inscribed equilateral triangle needs be bisected, and the bisecting line continued to the far side of the circumference of the circle. If then its point of intersection with the circumference be joined by a straight line to one of the base vertices of the triangle, it will be seen that the resulting line is one sixth of the entire circumference, or one side of a hexagon, and hence equal to the radius of the circle. From this recognition, through a series of simple logical steps, one rapidly arrives at the conclusion, i.e., the square on the side of the triangle is triple the square on the radius, and thus the proposition is demonstrated. Since Pacioli's slate is sharply foreshortened, the construc-

produced in the regular bodies treatise in Pacioli's *Summa de arithmetica, geometria, proportioni et proportionalità,* published at Venice in 1494. This small treatise was, like the painting itself, dedicated to Guidobaldo da Montefeltro. The *Summa,* a compendium of nearly 300 folios, is the book we see at the right, inscribed with Pacioli's name, for in 1495 it was his only printed work.

But Pacioli's publications did not come to an end with the *Summa.* His lessons, too, continue, and the right hand page of his copy of Euclid shows what lies ahead. Here the diagram indicates another theorem fundamental in the development of Pacioli's mathematics, Proposition 9, concerning a line divided according to the golden section, or « *divina proportione* »: « *Si latus exagoni equilateri latusque decagoni equilateri quos ambos unus idemq circulus circumscribit sibi invicem in longum directumq. Coniungantur totalinea ex eis composita s.m. proportionem habentem medium et duo extrema divisa erit maiorq. eius portio latus exagoni* » [18]. Modern geometers render Euclid thus: « If the side of the hexagon and that of the decagon inscribed in the same circle be added together, the whole straight line has been cut in extreme and mean ratio and its greater segment is the side of the hexagon » [19]. Illustrated in the margin of Pacioli's manuscript is a circle with

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tion of this line has been begun and is prominently displayed in the left forward segment of the circle, rather than in the far segment, nearest Pacioli, where it must first be constructed in order to make the proof. It has been displaced for the sake of clarity, and it is easy to see why, if one notes that all three segments defined by the inscribed triangle are, in objective terms, perfectly congruent, and hence of equal size. Subjectively, however, rendered perspectivally, the surface area of the far segment has become insufficient to permit the legible indication of the radius in its proper place.

[18] EUCLID, *Opera,* ed. Pacioli, fols. 124v-125r. PACIOLI teaches this again in the *Divina proportione,* Part I, Chapter 16

[19] *The Thirteen Books,* ed. Heath, III, 455-57.

the side of a decagon inscribed in it and, added to it, the side of a hexagon (equal to the radius of the circle).

The last illustrations on the manuscript folio before Pacioli represent the division of lines, and these undoubtedly relate to the lines shown on Pacioli's slate. Knowledge of the proportional ratios of both rational lines and lines divided according to the golden section was necessary for proposition 9 in Euclid's book, for the side of a decagon is an irrational line and the side of a hexagon is rational. Pacioli discussed linear proportions extensively in his commentary to Book V of the *Elements,* and he prefaced this book with the text of a lecture he gave on proportion in the church of San Bartolomeo in Venice in 1508 [20]. The division of lines into their mean and extreme ratios is, furthermore, the subject of propositions 1 through 6 of Book XIII. The lines illustrated on Pacioli's slate then play a distinct rôle in the larger theme of the painting, the Euclidean-Platonic solids.

In the lower left corner of Pacioli's slate, a column of square roots is summed, and square roots, or irrational numbers are required to solve exercises related to the golden section. Two of the numbers, 621 and 925, are remarkably close to numbers in the Fibonacci series, which in turn, approximate most closely the golden section proportion [21].

The golden section is also a prerequisite for constructing the dodecahedron, represented in perspective to the right. In turn, the dodecahedron is the one regular body which subsumes in its construction the other four. Thus it stands as a symbol of them all, and of Pacioli's lessons, for it unites the two principal motives we associate with his name, the regular bod-

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[20] EUCLID, *Opera,* ed. Pacioli, fol. 31r-v.

[21] The numbers in the Fibonacci series run: 1, 2, 3, 5, 8, 13, 21 ... 610, 987, 1597 ... See E. COLERUS, *Piccola storia della matematica da Pitagora a Hilbert* (Turin 1962), pp. 134-35.

ies and the « *divina proportione* », and it visualizes the immediate derivation of these themes in Piero's mathematics [22].

It was nearly fifteen years later, 1509, when Pacioli's *Divina proportione* was finally published in Venice. The manuscript, however, was completed in 1497 [23] and thus, at the time the Naples painting was executed, Pacioli was already composing it. The lessons of Pacioli's portrait, by linking the golden section and the regular bodies, anticipate the development from the *Summa* to the *Divina proportione.*

The most impressive feature of Pacioli's portrait remains to be noted. It is the shimmering, celestial crystal icosahexahedron, suspended to Pacioli's right, where, though immersed in light, it remains isolated by its singular perfection. Through the painter's perspective vision, it is at once both solid and hollow, in itself an extraordinary display uniting perspectival skills to the measurement of stereometrical solids. All its shining, transparent faces — eighteen squares and eight equilateral triangles — are simultaneously visible and reflect buildings, apparently the Palazzo Ducale in Urbino, from an unseen open window.

In his *Divina proportione* Pacioli describes the construction of just such an icosahexahedron. The chapter, number 53, is entitled « Dela formatione e origine del corpo del 26 basi, piano, solido, over vacuo, e delo elevato solido, over vacuo » [24] and illustrates a solid, hollow, and stellated example of this form (fig. 20). The text of the *Divina proportione* ex-

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[22] PACIOLI wrote in the *Divina proportione,* Chapter V, « *El quale ... senza la nostra proportione non e possible poterse formare.* » And in the same chapter he wrote concerning measurement of all five of the regular bodies: « *Li quali 5 regulari non è possible fra loro poterse proportionare. ne dala spera poterse intendere circonscriptibili senza la nostra detta proportione.* »

[23] PACIOLI, *Divina proportione,* fol. 23r.

[24] Ibid., fol. 15v., and table of contents.

plains that the two most important solids for architects are the icosahexahedron, or 26-faced solid, and the 72-faced solid, labelled by Pacioli « *Hebdomecontadissaedron* » (fig. 21). Of the 26-faced solid, Pacioli writes: « *E sia la sua scientia in molte considerationi utilissima a chi ben la acomodare maxime in architectura* » [25].

The detailed, closely focused mathematical programme of Pacioli's portrait was, no doubt, spelled out by Pacioli to his painter, who in turn must have known, or quickly learned, the mathematical concepts he was called upon to represent. Pacioli, an old hand with artists and an uninhibited pedagogue, clearly intervened in the painterly disposition of his ideas, for instance, in the studied positioning of the stereometric bodies in order to show their faces to best advantage, to cast their shadows most effectively, and to diminish most tellingly in perspective. And the realization in perspective of the rather intricate demonstration on Pacioli's slate evidences a direct collaboration between the painter and his intellectual mentor.

Pacioli apparently went far beyond even this in setting the confines of the visual programme. His *Summa*, published the previous year contains a small recurrent woodcut monogram (« LU ») incorporating the friar's portrait (fig. 16). In it, Pacioli, tonsured and wearing his Franciscan habit, stands before a table, pointing with a compass to the figures of a circle and a triangle in an open book before him, teaching as it were from the pages of the *Summa,* teaching, it almost seems, the same problem as in the Naples painting. This portrait simply expands the concept and format already established in the monogram. Pacioli's rôle in the design of the painting does

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[25] Ibid., fol. 15v. In the *Summa arithmetica* (part II, fol. 68v) Pacioli speaks proudly of the geometrical bodies he constructed for Guidobaldo da Montefeltro, Duke of Urbino; Pietro Valletari of Genoa, Bishop of Carpentras; and Giuliano della Rovere, the Cardinal of S. Pietro in Vincoli.

not, perhaps, end here. It is as if Piero had set his stamp on the design of Pacioli's stiff, heavy robe, which fills the center of the painting. Pulled in at the waist by a knotted cord, with large, weighty tubular folds moving from above and below toward the center (almost like lines drawn upon two intersecting cones), the robe repeats with clear understanding a common formula of Piero, a stylization that we meet in infinite variations turning the pages of any book on the painter. This stylization is little dependent upon the forms of the religious habit Pacioli wears, and it does not seem particularly congruent with the Naples painter's style, as far as it can be observed in this picture. It does, however, echo the monogram portrait of Pacioli, and the mathematician, himself, may have had in his possession a drawing, or some other record of his own likeness from the hand of Piero, inasmuch as Piero had portrayed him some twenty years earlier in the Brera altarpiece [26]. Certainly Pacioli required, or at least approved of this quasi-quotation of Piero's style, perhaps understanding it as a tribute to the recently deceased master whose ideas he propagated.

In any event, we must recognize Pacioli's responsiblity in the design of his own portrait. In the broadest possible

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[26] In his *Vite de matematici* (1589), B. BALBI describes a portrait of Pacioli in the Urbino Guardaroba by Piero della Francesca, which, however, is very possibly to be identified with the Naples work (BATTISTI, *Piero della Francesca,* II, 95, 257). F. HEINEMANN (*Giovanni Bellini e i belliniani* [Venice 1963]), I, 274-75, wrote that the « *ritratto di Fra' Luca fu ideato derivandolo dal prototipo di Piero della Francesca.* » M. MEISS's contention, reiterated by K. CLARK (*Piero della Francesca,* 2nd ed. [London 1969]), p. 68, that Pacioli, a Franciscan friar, could not conceivably have been represented as the Dominican Peter Martyr in the Brera Altar is not correct. The Brera painting was executed by 1474, and the first notice we have for Pacioli as a friar is in 1477. See STAIGMÜLLER, « Lucas Paciuolo. » p. 87; cf. A. FANTOZZI and B. BUGHETTI, « Il terz'ordine francescano in Perugia dal sec. XIII al sec. XIX, » *Archivium Franciscanum historicum,* XXXIII (1940), 357-58.

sense the invention is his. And in his hands the youthful, twenty-year-old painter was no doubt easily molded, and perhaps Pacioli hoped that under his tutelage, he would become a new Piero.

Who then is the painter of the work, or who is IACO. BAR. VIGENNIS, who painted it in 1495? When identified with a known artist, IACO. BAR. is always said to be Jacopo de'Barbari, although this identification has for many reasons, some of them rather futile, often been doubted by scholars. Serious chronological objections to the identification have, in recent years, been effectively dismantled by Creighton Gilbert, who also showed how intrinsically unlikely all the possible alternatives to the identification are [27]. Nonetheless, it is still sometimes felt that on stylistic grounds the painting is difficult to reconcile with the rest of Barbari's oeuvre.

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[27] C. GILBERT, « Alvise e compagni, » *Scritti di storia dell'arte in onore di Lionello Venturi* (Rome 1956), I, 290-92, n. 6; idem, « Ancora di Jacopo de' Barbari, » *Commentari,* VIII (1957), 155-56; idem, « When Did a Man in the Renaissance Grow Old? » *Studies in the Renaissance,* XIV (1967), 7-32; idem, « Barbari, Iacopo de', » *Dizionario biografico degli Italiani,* VI, 44-46.

Though the inscription shows signs of repainting, it is wrong to question its authenticity (HEINEMANN, *Bellini*, p. 274 and J. A. LEVENSON in *Early Italian Engravings from the National Gallery of Art* [Washington, D. C. 1973], p. 344, n. 15; cf. BATTISTI, *Piero della Francesca,* II, 95; A. DE RINALDIS, *Pinacoteca del Museo Nazionale di Napoli: catalogo* [Naples 1911, pp. 111-12]). The painting, acquired for Capodimonte in 1903, apparently never passed through the art market. A late, « spurious » inscription would not pin down the date of the painting so correctly, nor would it seek to ascribe the work to the then not so well known Jacopo de' Barbari, who had no known connection with Pacioli and whose paintings are not very distinguished, nor would it identify him so cryptically and with the unusual qualification « vigennis ». The puzzles of the inscription are authentic; they were not contrived by a later tamperer. Furthermore the lettering is consistent with that on the book as well as with Pacioli's own alphabet in the *Divina proportione.* Compare especially the rather distinctive R with a tail.

To answer this objection, we must take into account several characteristic aspects of Barbari's artistic personality and keep in mind a few peculiarities that we have noticed concerning Pacioli's portrait. Although Barbari has a number of distinctive traits, as an artist he did not always show a very strong or decided character. Moreover, he was an uneven, though sometimes quite distinguished performer; at times his works are eclectic and fragmentary from a stylistic point of view; finally he seems to have been extremely impressionable. Thus very many things may be possible in his paintings and indeed they are to be found in his autograph works alone. Yet, aside from the panoramic map of Venice, next to nothing he executed in Italy has been identified. And many of the works Barbari executed in the North are so filled with Germanisms that it was long believed he was not an Italian at all [28]. We cannot expect to find these elements in his Italian works.

On the other hand, we have seen Pacioli's large rôle in establishing the format of his portrait, and the prescribed, inevitable shapes of the geometrical objects further limit the sphere in which the style of « *Jaco. Bar. vigennis* » can manifest itself. Practically, one may look best to the portraits, the young man's costume, and the still-life objects.

In these circumstances, it is not really necessary to demonstrate a stylistic identity with Barbari's works executed after he left Italy. And rather than looking for a clear transition across this cultural divide, it is probably sufficient to find points of stylistic contact to add to the elements of the culture of Pacioli's portrait which recur in Barbari's later biography and art, and to add this to the common stylistic tradition from which both « *Jaco. Bar.* » and Jacopo de'Barbari are usually said to derive, a Bellinesque tradition — late Antonello, late

[28] LEVENSON, *Italian Engravings,* pp. 341-42.

Alvise Vivarini, late Quattrocento Venice. In doing so it seems correct to work, in part, with externals. Nonetheless bridges can be built across the Alps, and they need only confirm what is a securely signed and dated work.

Only a handful of signed, unquestionably authentic paintings belong to Jacopo — we can count only eight [29]. Nearly all of them show some variation on the quasi-Flemish realism of the Naples painting. It is also noteworthy that of these eight, six have the « *sfondo nero* » of the Naples portrait, a taste for dark backgrounds and silhouetting which appears in various guises in Jacopo's work. And, too, the bust or half-length format is that found in most of these works. But, there are more specific links in terms of details. Pacioli's pudgy, slightly clumsy hand with its poorly drawn nails reappears in Barbari's portraits of S. Osvaldo (1500; Private Coll., Hungary), that of Albrecht of Brandenburg (1508; Coll. H. Kisters, Kreuzlingen), and that of a Young Man in Vienna. The last of these portraits has apparently lost much of its surface, but the strong drawing of the eyes, the forceful modelling of the nose, and the rather full lips which seem lightly pressed upon the face are all immediately analogous to Barbari's possible self-portrait in Naples. These features are repeated, along with the heavily drawn brows, in the portrait of the Margrave Albrecht, where the note of coarseness common to the Vienna and Naples portraits is understandably muted. The thin-lipped, solemn likeness of Pacioli recurs in the portrait of S. Osvaldo, and the flesh of his face similarly disintegrates under the weight of his years. The same phenomenon is represented in a wholly different style in the signed Johnson panel (Philadelphia, 1503) and in the Louvre *Sacra Con-*

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[29] GILBERT, *Dizionario biografico*, pp. 44-45. For the portrait of Albrecht of Brandenburg, see the recent exhibition catalogue, D. KOEPPLIN and T. FALK, *Lukas Cranach*, I (Basel, 1974), no. 4.

versazione. « *Jaco. Bar.* » describes the corkscrew curls of the Naples youth with sets of concentric, circling gold lines, and Jacopo de'Barbari employs the same device in his bust of Christ in Weimar and in the St. John Baptist in his painting in the Louvre. Moreover, Barbari employs several quite distinctive configurations of draperies. It makes an amusing game to trace their reappearances in his paintings and engravings. The foldings of Pacioli's garment, as we have seen, need not be expected to appear among them, although the large, circling openings of his sleeves belong to a taste also present in the Louvre *Sacra Conversazione.* As an index of « *Jaco. Bar.* » in this respect, we must examine principally the sleeve of the standing youth. It happens that the close-fitted, crumpled sleeves of the Madonna in Jacopo's *Sacra Conversazione* now in the Berlin Museum (neither signed, nor dated, but always accepted) are formed in an identical pattern of alternating ridges and needle-eye depressions. This coincidence is not fortuitous, since it repeats itself in the Virgin of the Louvre painting, in a drawing of a male allegorical figure in the Uffizi, and in a few of Barbari's prints.

Jacopo de'Barbari's best known and most highly regarded painting, the Munich *Still-life* (1504), resembles Pacioli's portrait in Naples more closely that it does any other of Barbari's signed paintings. Its « *sfondo bianco* » is the aesthetic equivalent to the dark backgrounds of the Naples painting and of other works by Jacopo de'Barbari. Both are signed on an unfolded slip of paper, set realistically, *trompe-l'oeil,* in light and shadow. Both are concerned with textures and describing surfaces: the fur collar and feathers are treated with a similar touch and a similar sense for mottled effects, and the chain links of the mail match the metallic curls of the standing youth. All these realistic surfaces are infused with the same pristine light, and the still-life objects, whether a duck's foot or an inkpot, all cast carefully calculated shadows. The hard, metallic edges that exactingly outline the openings of the hood

and sleeves framing Pacioli's head and hands were created by a mind that would delight in the precise, steely circles of the metal armbands silhouetted against the shadows and the light ground of the Munich panel. The armor is, like the crystal icosahexahedron, a variation on a geometrically regular body, and the silvery facets of both are struck by sudden refracted light, often broken into reflected colors, which streak across the surfaces and which run along the edges. The same geometrical ordering of light and shade that defines the surface of the wooden dodecahedron in Naples forms the hunting spear in Munich, and the depiction of the wood grain of the dodecahedron is paralleled in the background of the Munich painting. Thus the luminous, optical, perspectival, geometrical culture that Pacioli's portrait proclaims, inconspicuously, but nonetheless surely, comes to fruition in the Munich still-life. This culture may also be traced in Barbari's graphic works, for instance, in his penchant for the bird's-eye view, à panoramic perspective; in the *scorci* of his Venice *Neptune,* his *Three Captives,* and his *St. Sebastian*; in the geometrical staging of his *Presentation, Guardian Angel,* and *Two Philosophers*; or finally in the crystalline celestial sphere in his *Apollo and Diana,* itself a graphic experiment with radiance, optical transparency, and eccentric perspective.

All these bridges that we have seen between « *Jaco. Bar.* » and Jacopo de'Barbari, are perhaps sufficient to allow us to look beyond the obvious stylistic differences that sometimes appear to divide them and to accept, without troubling second thoughts, the identification of the two artists as one.

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Whomever one sees behind the unfolded *biglietto* lettered IACO. BAR. VIGENNIS P. 1495, he knew Pacioli in 1495, he apparently traveled to Urbino and knew something of Piero's art, and, judging from his style, he was a Venetian, working in a late Quattrocento tradition. In any event, it is

Venice that draws artist and sitter together, and in the decade following 1495 everyone whom one would associate with Pacioli's portrait appears there.

In Venice Pacioli began the peripatetic career that led him back and forth across Italy — to Rome, Naples, Urbino, Milan, Florence, Pisa, Bologna, Perugia, and elsewhere. In 1464 he first came to Venice, staying for almost a decade in the house of the merchant Antonio de Ropiansi on the Giudecca as the tutor to the household's three sons [30]. During these years Pacioli followed the mathematics lessons of the Venetian Domenico Bragadino and others. And Venice was where he conversed about perspective with Giovanni and Gentile Bellini and with Hieronimo Malatini, another perspective teacher. In 1494, Pacioli returned to Venice to see his *Summa* through the press, and he seems to have remained there into 1495. And then, in 1508 and 1509 Pacioli was again in Venice to deliver lectures on the fifth book of Euclid, and to see his by then completed Euclid translation (1509) and *Divina proportione* (1509) published.

Even today Venice is small, and the world of artists and their admirers has never been very large. In these years Jacopo de'Barbari was in Venice, too, certainly from 1498 to 1500 [31], when he was preparing the great panoramic map of Venice for the German merchant Anton Kolb, and was thus engaged on the vast perspective and surveying enterprise that took its participants into every nook and cranny, down every alley way, and up the stairwells of every belltower of the city.

Dürer's quest after the secrets of Italian art drew him to Italy twice, both times in these years and both times to Venice. His first visit to Venice extended over 1494-1495, and his second from 1505-1507, the latter including his mysterious

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[30] STAIGMÜLLER, « Biographische Skizze, » p. 87.

[31] G. MAZZARIOL, T. PIGNATTI, *La pianta prospettica di Venezia del 1500 disegnata da Jacopo de' Barbari* (Venice 1962), p. 8.

trip to Bologna to learn the secrets of perspective [32]. During his second trip Dürer writes back to his friend Pirckheimer, that although now « there are many better painters here than Master Jacob », Anton Kolb still swears by him as the best painter in the world [33].

A few years earlier, in 1500, Barbari had been called to the court of Maximilian at Nuremberg, Dürer's native city, in the capacity of portraitist and miniaturist [34]. Like Pacioli in Italy, Barbari became in Germany a spokesman for the inclusion of painting among the liberal arts. Pacioli had argued that just as music was part of the Quadrivium, so also should perspective be [35] — and thereby painting, too, would become a liberal art. Jacopo de'Barbari pleads the same cause in a letter he addressed to Frederick the Wise, Elector of Saxony, around 1500. Painting, he wrote, should be declared the eighth liberal art, and hence put into the hands of noble men who would glorify it: « *...eser fata la otava arte liberale et eser levada de la mendicacione et eser posta in man de nobeli homeni, che la posi glorificare* » [36]. Because of its scientific foundations in arithmetic and geometry, painting belonged among the liberal arts — for there is no proportion without number, he wrote, and no form without geometry:

> E senza ese arte non pol essere pitura probabile fata da pitori, se non sarà periti nelle sopradite arte, prima nella giometria, po' aritmetica, lequal due necessita nelle commensuracione de proportione, ché non pol essere proporcione senza numero, né pol essere forme senza giometria [37].

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[32] E. Panofsky, *Albrecht Dürer,* 3rd ed. (Princeton 1948), I, pp. 8-9.
[33] *Dürer, Schriftlicher Nachlass,* ed. H. Rupprich, I (Berlin 1956), 44.
[34] L. Servolini, *Jacopo de' Barbari* (Padua 1944), p. 74, n. 20: « *contrafeter und illuminist.* »
[35] Pacioli, *Divina proportione,* fol. 3r.
[36] P. Barocchi, *Scritti d'arte del Cinquecento,* I, (Milan 1971), 69.
[37] Ibid., pp. 66-67.

Barbari's concern with perspective and proportion and his attempts to provide the art of painting with a theoretical and scientific foundation were certainly reasons why Albrecht Dürer was interested in him. Possibly the two men met as early as 1495, during Dürer's first stay in Venice. In any event, in a draft, written in 1523, of the dedication of his *Proportionslehre* to Wilibald Pirckheimer, Dürer describes his meeting with Jacopo, his introduction to the scientific aspects of Barbari's art, and the catalytic effect Barbari had on his own theoretical studies.

> Dan ich selbs wolt lyber ein hochgelerten berumbten man jn solcher kunst hörn und lesen, dan das ich als ein unbegrünter dofan schreiben söll. Jdoch so ich keinen find der do etwas beschriben hett van menschlicher mas zw machen dan einen man Jacobus genent, van Venedig geporn, ein libricher moler. Der wies mir man und weib, dy er aws der mas gemacht het, und das ich awff dyse tzeit liber sehen wolt was sein mainung wer gewest dan ein new kunigraich, und wen ichs hett, so wolt jch ims zw eren jn trug pringen, gemeinem nutz zw gut. Aber ich was zw der selben tzeit noch jung und het nie fan sölchem ding gehört. Und dy kunst ward mir fast liben, und nam dy ding zw Sin, wy man solche ding möcht zw wegen pringen. Dan mir wolt diser forgemelt Jacobus seinen grunt nit klerlich tzeigen das merckett ich woll an jm. Doch nam ich mein eygen ding für mych, und las den Fitrufium, der beschreibt ein wenig van der glidmas eines mans. Also van oder aws den zweien obgenanten manen hab ich meinen anfang genumen, und hab dornoch aws meinem für nemen gesucht van dag zw dag [38].

Although Dürer was unable to extract from Barbari his secrets — his canon of proportions and the principles on which it was based — he could turn to ancient and modern published

[38] Dürer, *Nachlass*, I, 102.

sources, such as Vitruvius, Euclid, and Pacioli. Prompted by Barbari's example, aided by the humanist library of ancient works belonging to his friend Pirckheimer, and furnished with the printed treatises of modern Italian theoreticians, Dürer composed his own treatises on measurement and proportion for German artists.

In his projected manual for the education of the artist, *Der Speis der Malerknaben,* Dürer hoped to train aspiring painters in the foundation of art, measurement and proportion. The dedication of his *Unterweisung der Messung* (Nuremberg 1525), the first part of the never completed *Speis der Malerknaben,* spells out Dürer's fundamental conviction that the art of measurement was the basis of painting.

> Die weyl aber die [Kunst der Messung] der recht grundt ist aller mallerey, hab ich mir fürgenomen allen künstbegyrigen jungen, eyn anfang züstellen, und ursach zügeben damit sie sich der messunge zirckels und richtscheyt underwinden, unnd darauss die rechten warheyt erkennen unnd vor augen sehen mögen, damit sie nit alleyn zü künsten begirig werden, sonder auch zu eynem rechten und grösseren verstant komen mögen [39].

The fourth and last book of the *Unterweisung* shows clear connections with the work of Dürer's Italian « masters ». This book treats polyhedra, and Dürer writes that these are of three kinds: first, the elongated and cylindrical (e.g., towers and columns); second, the pointed (e.g., pyramids); and third, the regular bodies, symmetrical in every respect [40].

Dürer's definitions recall those of Piero's *Libellus.* Dürer

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[39] A. Dürer, *Underweysung der messung mit dem zirckel und richtscheyt* (1525; facsimile rpt. Dietikon-Zürich 1966), verso of title page.

[40] Book IV, n. pag., see diagram 29.

describes them as « *Corpora die allenthalben gleych sind, von felderen, ecken und seyten, die der Euklides corpora regularia nennet, der beschreibt jr fünffe, darumb das jr nit mer künnen sein, die in ein kugel darin sie allenthalben an rüren verfasst mügen werdenn, die selben nach dem sie zü vill dingen nutz sind, wil ich hie anzeygen* » [41]. Piero, in his *Libellus,* had defined the « *corpora* »: « *Multa sunt corpora lateribus constituta quae in sperico corpore locari queunt, ita ut eorum anguli sperae superficiem omnes contingunt. Verum quinque ex eis tantummodo sunt regularia: hoc est, quae aequales bases habent et latera* » [42].

Piero had illustrated his *Libellus* and the *Trattato d'abaco* with fine pen drawings of the regular bodies in perspective, executed with a compass and ruler (figs. 17, 18, 19). Pacioli, when he published Piero's treatise in the *Divina proportione,* relied on large woodcuts depicting models of solid and hollow geometrical bodies, drawn in perspective (figs. 20, 21). Dürer carried this process of illustration one step further. Instead of showing the regular and irregular bodies foreshortened, he provided two-dimensional diagrams as patterns for constructing from paper each of the five regular Euclidean bodies, nine of the irregular Archimedean bodies, and the sphere. With these patterns the construction of a set of geometrical bodies became a simple matter: « *Aus disen dingen* », he wrote, « *mag man gar manicherley machen, so jr teyl aufeinander versetzt wirt, das zü dem aushauen der seulen und jren zirden dinet* » [43] (fig. 23).

A knowledge of these solids not only had useful applications in architectural matters, but it had concrete practical advantages for painting. And so, Dürer proceeds to demon-

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[41] Ibid.

[42] PIERO DELLA FRANCESCA, *L'opera* « *De corporibus regularibus,* » ed. G. Mancini (Rome 1916), p. 489.

[43] DÜRER, *Underweysung,* Bk. IV, n. pag., diagram 43.

strate how to place them in a painting. « *So ich daforen manicherley corpora wie man die mach anzeigt hab, wil ich auch leren so man soliche gemecht ansicht wie man die in ein gemel müg pringen...* » [44]. The example he chooses to illustrate is the cube (fig. 24), showing how to represent it in correct perspective and how the six-faced regular body will cast shadows under specific light conditions » [45].

Following his experiments with light and shadow, Dürer proceeds directly to the representation of the cube in perspective giving two methods that rely on the intersection of the visual pyramid [46]. These belong to the scientific demonstrations of « *perspectiva communis* », i.e. mathematically worked out solutions rather than approximated ones. « *In gleycher weyss wie ich den cubus in ein abgestolen gemel gebracht hab* », Dürer writes, « *also mag man alle corpora die man in grund legen und aufziehen kan, durch söliche weg in ein gemel pringen* » [47]. This statement is crucial to the development of his later *Proportionslehre,* for his particular model of the cube can be generalized, and all ordinary solid objects can be rendered in a geometric manner. Dürer then closes the *Unterweisung* with artificial means of measurement for painters, the « *perspectiva artificialis* » solutions, methods of seeing through a framed glass with one eye in a limited, fixed position, and for establishing the composition by means of threads running from the object to the wall in front of it, intersecting a small door.

With his patterns for constructing regular and irregular solids and with his explicit diagrams for representing them foreshortened and under specific conditions of light, Dürer

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[44] Ibid., verso of diagram 51.
[45] Ibid., diagrams 52, 56, 57, 58.
[46] Ibid., diagrams 59, 60, 61, 62.
[47] Ibid., diagram 62.

brought the theme of the Euclidean bodies into direct relationship with the concerns of the practicing artist. The Italians, Piero and then Pacioli, had established the relationship of the regular bodies to the artist and architect on a theoretical basis, but the practical implications were less explicitly demonstrated. In the *Trattato d'abaco* the regular bodies were a component of the broader topic of solid geometry, but Piero's *Abaco* demonstrations were nonetheless close to those of his later *Libellus*. These complex mathematical schemes of measurement showed the proportional relationship of regular bodies to a sphere and to one another, and thus established the theoretical basis for their construction. An artist who wanted to further the perspectival skills he acquired from the *Prospectiva pingendi* could sharpen them by following the mathematical demonstrations in the *Libellus*; in this way, too, his perspectival knowledge would be set on a firmer mathematical foundation. For Pacioli, the regular bodies took on a Platonic, and quasi-mystical significance. Thus their consideration, or contemplation, was an affair undertaken far above the ordinary level of everyday discourse. Because of the necessity of employing the golden section in constructing the dodecahedron — the Platonic symbol of the Universe which subsumed the other four regular bodies — all the regular solids were called « divine » and thought to have *effetti* ranging from *dignissimo* and *excellentissimo* to *inextimabile* and *mirabile* [48]. Dürer in a sense brought the topic down to earth again by teaching aspiring artists in a practical way how to construct regular and irregular bodies from paper and how to use them for exercises in perspective and in light and shade. His principal objective was to teach, through measurement, the correct representation of solid objects in space, and since all objects could be treated

[48] PACIOLI, *Divina proportione,* Part I, fols. 5v-7v.

according to the model of the cube, the practical implications of the Euclidean bodies became infinitely more numerous [49].

Not only could all solid inanimate objects found in nature be rendered scientifically, and hence correctly, but in his *Proportionslehre,* Dürer even treated anatomical renderings in geometrical terms. Book IV, dedicated to figural movement, functions as a postscript to Dürer's geometrical studies in the *Unterweisung der Messung* [50]. First he shows how to foreshorten faces by means of the « third outline » — a procedure he perhaps learned from Vincenzo Foppa's lost perspective manual (fig. 25) [51], but one also closely related to Piero della Francesca's head and capital studies in the *Prospectiva pingendi* (figs.

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[49] The relationship between the art of lettering and the study of measurement and proportion is made clear in Pacioli's directions in the *Divina proportione* for forming the letters of the alphabet. In his *Unterweisung der Messung* DÜRER similarly shows how to measure and form the letters of the Latin as well as the Gothic (*Fraktur*) alphabet. His instructions are intended for « builders and also painters, and those who need to place letters high up on a wall » (Bk. III, n. pag., see Latin letter A). A somewhat later pictorial source shows that humanist letter designers studied the regular bodies. Nicolaus Neufchatel's painting (1561) in the Alte Pinakothek, Munich, shows Johannes Neudörfer, the famous Nuremberg mathematician and calligrapher, teaching his son how to measure a hollow dodecahedron with a compass. Above them hangs a hollow cube, and around the frame of the canvas runs an inscription which is a masterpiece of lettering. Neudörfer was the first to write a set of artist biographies in the north, and in his work he attached a high value to the study of Euclid's *Elements* (*Nachrichten von Künstlern und Werkleuten* [Vienna 1875], p. 48). For Neufchatel's painting see R. A. PELTZER, in *Münchner Jahrbuch der bildenden Kunst,* III (1926), 192-94; *Alte Pinakothek München: Kurzes Verzeichnis der Bilder* (Munich 1957), p. 73, plate 118; *Deutsche und niederländische Malerei zwischen Renaissance und Barock,* Alte Pinakothek München, Katalog I (Munich 1961), p. 36. A follower of Neufchatel painted Stephan Brechtel, also a mathematician and calligrapher, teaching his son the characteristics of an irregular polyhedron. Above them is suspended a hollow sphere (PELTZER, p. 195).

[50] A. DÜRER, *Vier Bücher von menschlicher Proportion* (1528; facsimile rpt. Unterschneidheim 1969), also called the *Proportionslehre.*

[51] E. PANOFSKY, *Dürers Kunsttheorie* (Berlin 1915), pp. 54-55.

26, 27). From frontal and profile views it is shown how to find the « ground plan », as it were, or the third outline, of the whole head. And from a profile inscribed in a rectangle tipped upwards or downwards, Dürer demonstrated how to foreshorten the head seen from above and below. Utilizing the « plans » of heads and whole bodies, the human figure may be ascribed the same properties as the inanimate cube, and thereby, in theory, it may be studied, outlined, and projected onto the picture surface as a regular body.

Dürer continues, in the last book of the *Proportionslehre,* demonstrating the enclosure of limbs and other parts of the body within plane and solid geometrical figures (fig. 28). These figures, we recognize, are simplifications of the regular and irregular solids of the *Unterweisung,* and consequently the procedure Dürer outlines aids in understanding the volumes of the human body and its correct foreshortening. There exist drawing after drawing by Dürer to chart his concern with geometricizing human figures, thus rendering them scientifically (figs. 29, 30)[52]. In their application to human figures the knowledge of abstract geometrical bodies is brought entirely into the realm of painting[53].

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[52] See esp. *Das Skizzenbuch von Albrecht Dürer in der Königl. Öffentl. Bibliothek zu Dresden,* ed. R. Bruck (Strassburg 1905). Also W. STRAUSS, *The Complete Drawings of Albrecht Dürer* (New York 1974), V, HP 1527/13; HP 1527/19; HP 1527/21; HP 1527/25 and passim. For Dürer's drawings showing the use of regular and irregular bodies in goldsmith designs, see STRAUSS, IV, 1526/12-1526/22. For polyhedron sketches, VI, AS 1514/19; AS 1514/20; and passim. In addition for Dürer's drawings of human figures proportioned according to the « progressive proportion » see STRAUSS, V, HP: 1513/7; HP 1513/10; HP: 1513/14 HP: 1513/16; HP: 1513/18.

[53] Such elaborations in the application of pure geometrical solids to the representation of the human figure are related to the small wooden models, with moveable limbs, that German artists drew. For the literature on these mannequins, see A. WEIXLGÄRTNER, in *Beiträge zur Kuntsgeschichte Franz Wickhoff gewidmet* (Vienna 1903), pp. 80-90; J. v. SCHLOSSER, in *Jahrbuch der Kunsthistorischen Sammlungen des Allerhöchsten Kaiser-*

In 1568, a book with a long, descriptive title appeared in Nuremberg, the goldsmith Wenzel Jamnitzer's *Perspectiva corporum regularium, das ist ein fleyssige Fürweisung wie die Fünff Regulirten Cörper, darvon Plato inn Timaeo, unnd Euclides inn sein Elementis schreibt etc., durch einen sonderlichen, newen, behenden unnd gerechten Weg... gar künstlich inn die Perspectiva gebracht... werden mögen* »[54]. Jamnitzer's book represents the culmination of the researches of Piero, Pacioli, and Dürer, for it illustrates in perspective, not only the five regular bodies, but also over one hundred and fifty other geometrical solids deriving from the five fundamental forms (figs. 31). Owing to the great variety of geometrical bodies presented and to the high quality of Jost Amman's engravings, Jamnitzer's *Perspectiva corporum regularium* may deservedly be considered a virtuoso performance. Yet it represents, in another sense, a simplification of the tradition Piero, Pacioli, and Dürer established. One indication of this

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hauses, XXXI (1913-14), 111-18; B. DEGENHART, in *Zeitschrift für Kunstgeschichte,* VIII (1939), 125-35; A. WEIXLGÄRTNER, in *Årstryck, Göteborgs Kunstmuseum* (1954), pp. 37-71. For Piero della Francesca and the use of models see VASARI-MILANESI, II, 498-99; J. v. SCHLOSSER, « Künstlerprobleme der Frührenaissance, II, Piero della Francesca, » *Akademie der Wissenschaften in Wien, Philosophisch-historische Klasse, Sitzungsberichte,* 214. Bd., 5 Abh. (1933), 10; E. BATTISTI, *Piero della Francesca,* I, 483, n. 187. From these *Possen,* or mannequins (which DÜRER specifically mentions in his *Proportionslehre*), and from Dürer's studies in the measurement of abstract geometrical bodies and their application to the representation of movement in the human figure, another type of artists' manual was developed in the north. It is best exemplified by ERHARD SCHOEN'S *Underweysung der Proportion und Stellung der Bossen* ... (Nuremberg 1538) and H. LAUTENSACK'S *Des Cirkels und Richtscheyts* ... (Frankfurt 1564). For these works see J. v. SCHLOSSER, *La letteratura artistica* (Florence 1964), pp. 278-80.

[54] Facsimile ed. Graz 1973. See also W. JAMNITZER, *Orfèvre de la rigueur sensible: étude sur la « Perspectiva corporum regularium, »* ed. A. Flocon (Paris 1964). For the relationship of regular bodies to goldsmith practices; K. PECHSTEIN, « Jamnitzer-Studien, » *Jahrbuch der Berliner Museen,* VIII (1966), 239.

is that Jamnitzer fails to provide the reader with directions for costructing geometrical bodies. Moreover, little or nothing of the mathematical or proportional theory which lies behind the regular solids finds its way into the pages of Jamnitzer's work. Instead the geometrical forms have become academic stereometric exercises for the study of perspective. Jamnitzer's work must be considered a kind of pattern book, or copybook, to be studied, perhaps, in conjunction with oral instruction. Nonetheless, one Italian theoretician, who knew the history of the subject, recognized the true origins of Jamnitzer's book. This theoretician was Egnazio Danti, who wrote: « *Wenceslao Giannizzero Norinbergense, il quale hà messi in Prospettiva li corpi regolari, et altri composti, si come fece Pietro dal Borgo, se bene Fra Luca gli stampò sotto suo nome* » [55].

Not long after the publication of Dürer's *Unterweisung der Messung* and his *Proportionslehre*, each was translated from German into Latin, thus becoming more generally accessible to the flourishing community of Italian scholars and theoreticians of the Cinquecento [56]. One of the most learned of these was Daniele Barbaro, Patriarch-Elect of Aquileia (1514-1570), Venetian ambassador to the Council of Trent, mathematician, astrologer, and inventor [57]. One of the most prolific and many-sided treatise writers of the sixteenth century, Barbaro reveals, in his perspective works, a knowledgeable command of Piero's ideas. Barbaro's deep familiarity stemmed in part from a direct acquaintance with Piero's work, but his knowledge

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[55] J. Barozzi Vignola, *Le due regole della prospettiva pratica*, ed. E. Danti (Rome 1583), Preface.

[56] Schlosser, *Letteratura artistica*, pp. 273-74.

[57] See G. Alberigo's entry in *Dizionario biografico degli Italiani*, VI, s.v. « Barbaro, Daniele »; *Andrea Palladio 1508-1580*, ed. H. Burns, Arts Council Exhibition (London 1975), pp. 94-96.

of Piero derived also in part from the writings of Pacioli and those of Dürer. Barbaro himself wrote two perspective manuals. Their titles are practically identical, though one was published and the other remained in manuscript. The first, *La pratica della perspettiva,* was published in Venice in 1569. It contains nine chapters on various aspects of perspective science, with frequent references to Piero della Francesca. The title of Barbaro's second, «La pratica della prospettiva,» varies by a single letter. Nonetheless it is a wholly separate treatise. It exists in manuscript in the Biblioteca Marciana in Venice, and it is precisely a treatise on the regular and irregular bodies [58].

Barbaro's marked interest in perspective and regular bodies, as he himself relates, initially grew out of his Vitruvian studies. He, like Vitruvius, refers to perspective as « *scenographia* » [59]. His published treatise on perspective, *La pratica della perspettiva,* leads from detailed instructions concerning polygons and polyhedra to the correct painted representations of buildings and their architectural parts to the end of constructing theatrical scenery. Chapters I and II of the *Perspettiva* teach the principles of perspective and the geometrical construction and foreshortening of polygons; Chapter III, by far the longest, the « *modo di levare i corpi dalle piante* », e.g., the way to construct the three-dimensional dodecahedron from its two-dimensional plan, the pentagon; Chapter IV concerns « *scenographia* ».

Barbaro's discussion of the regular and irregular solids in Chapter III of the *Perspettiva* is far more exhaustive than

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[58] D. Barbaro, *La pratica della perspettiva* (Venice 1569), and idem, « La pratica della prospettiva, » Venice, Bibl. Marciana, MS. It. IV, 39 (=5446). The ms. of the published book is also in the Marciana, MS. It. IV, 40 (=5447).

[59] Barbaro, *Perspettiva* (1569), p. 3; idem, « Prospettiva, » MS., fol. 298r; Vitruvius, *De architectura* (VII, Preface, 11).

anything we have seen before in Piero, in Pacioli, or even in Dürer. It is Barbaro who, besides giving patterns for the bodies and instructions in shading, gives explicit and very detailed directions for rendering most of the regular and irregular bodies in perspective with a compass and rule. His instructions for constructing the dodecahedron illustrate well the detailed character of his treatment.

> Io ponerò qui appresso quello, che appartiene alla pratica. Et perche tutti questi corpi regulari sono circonscritti dalla sphera, cioè con tutti gli anguli loro toccarebbero la concavità d'una sphera nella quale fussero rinchiusi, però nella formazione delle loro piante perfette, si formano in uno circolo. Facciasi adunque sopra 'l centro a uno circolo, & sia partito in dieci parti equali b, c, d, e, f, g, h, i, k, l, e alternamente sopra quelle parti facciansi due soperficie di cinque lati equali, l'una sia bdfhk, l'altro cegil, etc (fig. 32)[60].

Drawing regular and irregular bodies in perspective was evidently, for Barbaro, an exercise in learning how to render buildings. In stage design, that is, in « *scenographia* », there is but a single vanishing point for the painted architecture, and the individual components must be established with reference to this point. Starting with the relatively simple regular bodies, one proceeds to complicated irregular solids, and then ultimately to an accurate, correct perspectival rendering of columns, bases, capitals, architraves, portals, vaults, and other architectural components (fig. 33). Moreover, Barbaro's demonstrations show how to shade these elements. Thus the study of abstract polygons and polyhedra was, in Barba-

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[60] BARBARO, *Perspectiva* (1569), pp. 49-50. Cf. DÜRER, *Underweysung*, Bk. IV, diagram 33, where the overlapping pentagons are shown but where no verbal instructions are given. Danti's commentary to VIGNOLA'S *Due regole*, p. 143, mentions the lack of instruction in JAMNITZER'S treatise: « *se bene hà delineate solamente le figure senza scrivervi attorno cosa nessuna.* » One may imagine Barbaro's directions were welcome aids.

ro's published *Pratica della perspettiva,* intended as training in the correct representation of objects found in nature, and in this way the usefulness of understanding regular and irregular bodies is brought to bear on both painting and architecture.

Barbaro's Venetian manuscript, « La pratica della prospettiva », directly relates the study of geometrical bodies to perspective, and hence to the practical concerns of artists: « *conviene che hora dimostramo alcuna delle forme irregolari, perche oltra che elle sono di maggior numero et molto più pertinenti al dipintore che le regolare* » [61]. Entitled « Prospettiva », Barbaro's treatise contains a concise introduction to the principles of perspective, but primarily, the treatise is dedicated to the regular and irregular bodies.

The knowledge of perspective, Barbaro writes, is a prerequisite to understanding the treatise at hand. In a treatise not for beginners, he recapitulates the principles of perspective explained at length in his published book. Barbaro loses little time arriving at the subject which truly interests him, the construction and perspectival rendering of geometrical bodies. As in his published book on perspective, here, too, Barbaro explicitly connects the ability to render these stereometric solids with skills in rendering architectural motifs.

> Hor perche gli e necessario volendo per arti di prospettiva disignari sopra uno quadro qualche architettura, over altra cosa che stai debbia, come essa perspettiva richiede, intendo dir il suo perfetto, non dico perfetto quanto a tutta la forma: perche presupono che uno dipintor, oltre la perspettiva debba perfettamente intendere quali siano le buone forme: si di gli edificij, come da tutte le altre cose: volendo nel arte sua procedere secondo che ella richiede: ma solamente intendo per il suo perfetto haver cognitione che forma, che li habbia, cio è triangolare, quadrata, pentagona, over di altra sorte: et come

[61] BARBARO, « Prospettiva », » MS., fol. 19r.

> tutti li anguli di essa architettura, over altra cosa essendo materialmente costruita [62].

On folio 12r Barbaro introduces the ways of foreshortening polygons, and, on folio 25r he continues with instructions in constructing,in shading, and in rendering perspectivally the regular and irregular solids. The figures are not always drawn in, but their designs are incised by the compass points into the paper, and hence ready to be inked in. Patterns for an immense variety of geometrical solids are given, and examples of geometrical solids inscribed in others are shown. From the more or less simple geometrical bodies, there are developed elevated and stellated regular bodies and intricate *mazzocchi* (figs. 34-36). A whole section, extending from folio 168r through 215r, is devoted to the sphere, and this theme, too, is brought specifically into the painter's domain: « *...et se un pittore volesse far in pittura un corpo spherico, over balla, tirra col compasso un circolo, et quello date le sue ombre diligentemente havrà fatto lo detto corpo spherico* » [63].

Most of the treatise, approximately 270 folios of a total of 310, is devoted to the correct perspectival drawing of geometrical bodies. Barbaro admits the difficulty of rendering the

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[62] Ibid., fol. 7v.

[63] Ibid., fol. 168r. A related exercise in PIERO'S *Prospectiva pingendi* (p. 210) seems to have been overlooked in all the literature on the egg in the Brera panel (most recently: I. RAGUSA, « The Egg Reopened, » *Art Bulletin,* LIII [1971], 435-43): « *Acade al le volte de volere dimostrare sopra de alcuna taula o spazzo, o socto a sularo, alcuno corpo o sopra o socto a quelli posto, siccomo sopra delli spacci tu volesse circulare et contorneare corpi che paressero elevati, cioè casse, deschi, palle, animali et similmente sopra taula da mangiare vasi, candelieri e altri corpi, così socto sulari o socto volte anelli o altre cose che pendessero, che ad termine paressero commo veri.* » Piero was interested in the correct representation of solids suspended from a vault. Although he, like Barbaro, was concerned mainly with circles and spheres, there are instructions for representing the ovoid form in DÜRER (*Underweysung*, Bk. I, diagrams 22, 34). Still another egg in F. Benaglio's *Sacra conversazione*, S. Bernardino, Verona, and in Ercole de' Roberti's *S. Girolamo*, S. Petronio, Bologna.

most complicated of these forms, but he nevertheless urges their study, even if the student must read his book « *quattro, et sei volte con grande difficoltà et non senza gran fatica* » [64]. The work, in fact, bespeaks a fascination with these stereometric solids similar to that of Luca Pacioli. But unlike Pacioli, Barbaro's discussions are without philosophical or metaphysical speculations. Instead, for the Venetian, the study of regular and irregular polyhedra afforded artistic exercises in developing a hand practiced in the accurate, yet easy rendition of three-dimensional space and stereometrical volume.

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The measurement of regular and irregular bodies, which Piero initiated, became in time a standard feature of the perspective manuals used by artists and architects [65]. This stereometrical mensuration was dependent upon the fixed ratios existing among the Euclidean solids and the sphere. And, it was a knowledge of these invariable proportions that permitted the accurate measurement of abstract polygons and polyhedra.

In the foregoing remarks concerning the legacy of Piero in the work of Pacioli, Dürer, and Barbaro, we have concentrated on the relationship of the regular bodies to perspectival study. Perspective was emphasized in order to underline

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[64] BARBARO, « Prospettiva, » MS., fol. 252v.

[65] Besides those noted, see also: COSIMO BARTOLI, *Del modo di misurare* (Venice 1564), fols. 83-91; JEAN FRANÇOIS NICERON, *La perspective curieuse, ou Magie artificielle* (Paris 1638), Book I, pp. 28-36, and illustrations; GIOVANNI AMICO, *L'architetto prattico* (Palermo 1726-50), II, 141, figs. 21, 22; *Elementi di perspettiva secondo li principii di Brook Taylor con varie aggiunte ... del Padre Francesco Jacquier* (Rome 1755), pp. 46-47, figs. IX, X; BERNARDO ANTONIO VITTONE, *Istruzioni elementari per indirizzo de' giovani allo studio dell'architettura civile* (Lugano 1760), pp. 29-30; EUSTACHIO ZANOTTI, *Trattato teorico-pratico di prospettiva* (Bologna 1766), p. 75 ff., figs 36-42; FERDINANDO GALLI BIBIENA, *Direzioni a' giovani studenti nel disegno dell'architettura civile* (Bologna 1777), Preface, pp. 30-36, and illustrations.

the relationship between Piero's *Abaco* and *Libellus* and his perspective treatise. But the regular bodies have many faces, and they exercised yet another fascination over Piero's follower, Luca Pacioli, whose enchantment with proportion is epitomized in the title of his treatise « *divina proportione* ». Thus proportion, rather than perspective, could have been our theme. Within the realm of the regular bodies, perspective and proportion are often but two sides of the same concern. Much of what we discover in reading Piero's mathematical works is just this measuring of regular bodies, set one within the other and within a sphere, through proportional relationships derived from Euclid's theorems. In turn, Piero applied numbers to these proportional relationships, thereby deriving a system of proportional measurement.

Along these lines, Pacioli's statements concerning proportion lay bare the underlying congruity of Piero's three treatises. Proportion, writes Pacioli, is the mother and queen of all the arts — none of them can be practiced without it (« *Tu troverai la proportione de tutte esser madre e regina: e senza lei niuna poterse exercitare* »). As a part of perspective, proportion will determine the size of a human figure both in relation to its setting and in terms of its distance from the viewer. And proportion, too, will determine where a figure should be placed in the scheme of the entire painting [66]. All this, Pacioli continues, Piero has demonstrated in his *Prospectiva pingendi* where « *de le 10 parole le 9 recercano la proportione* ».

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[66] PACIOLI, *Summa,* fol. 68v. « Questo el prova prospectiva in sue picture. Le quali se ala statura de una figura humana: non li da la sua debita grosseçça negli occhi de chi la guarda: mai ben responde ... E cosi nelli piani dove hano apostare tal figura: molti li convene haver cura: de farla stare con debita proportione de distantia ... E cosi facendo una figura a sedere sotto qualche fornice: bisogna che la proportionino in tal modo: che levandose in piedi: ella non desse del capo fore coperchio. E cosi in altri liniamenti e dispositioni de qualunche altra figura si fosse.

Appendix I

Luca Pacioli and the «Trattato d'abaco»

Piero della Francesca's mathematical pursuits constitute an integral part of his biography by Vasari. From his youth, Vasari relates, Piero applied himself to the mathematical sciences, and even after he turned to painting in his fifteenth year he never ceased studying them — in fact, he made « *maraviglioso frutto ed in quelle e nella pittura* ». Learned in perspective and highly knowledgeable concerning the secrets of Euclid, Piero understood the properties of geometrical bodies better than any master of his time. The most useful elucidations of these matters which we possess, Vasari tells us, are from his hand. Owing to his writings, Piero merits being called the « *miglior geometra che fusse nei tempi suoi* » [1].

The contributions of Piero *matematico* went unrecognized, Vasari continues, because, after Piero's death, the fruits of his labors were published under the name of his compatriot, Fra Luca Pacioli of Borgo San Sepolcro: « *e venuto Piero in vecchiezza ed a morte, dopo avere scritto molti libri, maestro Luca detto, usurpandoli per se stesso, gli fece stampare come suoi, essendogli pervenuti quelli alle mani dopo la morte del*

[1] VASARI-MILANESI, II, 487-501.

maestro »[2]. Vasari neglected, however, to support his contention by specifying which of Piero's books Pacioli had usurped. Since, with the passage of time and the multiplication of printed works treating geometry, perspective, and the regular bodies, most of Piero's manuscript writings were lost sight of, Vasari's allegation of plagiarism long persisted as an issue, mainly in commentaries on Vasari's *vita* of Piero. Nonetheless, there was little concrete evidence to bring to bear upon the question.

Then, in 1880, the German scholar Max Jordan uncovered Piero's *Libellus de quinque corporibus regularibus* among the Urbino manuscripts in the Vatican Library (Vat. Urb. lat. 632). By comparing Piero's Vatican *Libellus* with Luca Pacioli's « Libellus in tres partiales tractatus divisus quinque corporum regularium et dependentium », a treatise dedicated to Piero Soderini, appended to Pacioli's *Divina proportione,* and published in Venice in 1509, Jordan recognized that Pacioli's « Libellus » on the five regular bodies was only a direct Italian translation of Piero's Latin *Libellus* in the Vatican[3]. Thereby Jordan was able to confirm that Vasari's allegation of Pacioli's plagiarism was substantially correct.

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[2] Ibid., p. 498. Precisely at the time VASARI was preparing the first edition of his *Lives,* in 1546, a clamorous case of plagiarism came to the fore in Florence. Perhaps significantly, the issue at hand was a mathematical one, the solution of the cubic equation (see above, Chapter II, p. 23 f.). Its two main protagonists, Niccolò Tartaglia and Gerolamo Cardano, marked their dispute indelibly in their longer publications and with a flurry of short polemical tracts that spread throughout Italy. It seems highly possible that this controversy played a rôle in Vasari's formulation of Piero's *vita.*

[3] M. JORDAN, « Der vermisste Traktat des Piero della Francesca über die fünf regelmässigen Körper, » *Jahrbuch der Königlich Preussischen Kunstsammlungen,* I (1880), 112-18.

For the question of Pacioli's plagiarism and the literature leading up to the discovery of the *Libellus,* see the editions of the *Lives* edited by G. DELLA VALLE (III, 1971, 248n); G. ANTONELLI (V, 1828, 14n); G. MASELLI (1832-38, p. 297, n. 2); L. SCHORN (V, 1837, 297, n. 4); F. RA-

It has not been noted, however, that Pacioli's treatise on the measurement of the five regular bodies, appended to the *Divina proportione* in 1509, was preceded by a smaller treatise of some eleven pages on the same subject. Entitled « Particularis tractatus circa corpora regularia et ordinaria », and dedicated to Guidobaldo da Montefeltro, Duke of Urbino, it constitutes the final chapter of Pacioli's *Summa de arithmetica, geometria, proportioni et proportionalità,* published in Venice in 1494 [4]. Pacioli's introduction to the work is informative and bespeaks his own genuine fascination with the subject.

> E benche di sopra in questo nella distintion .6ª. al capitolo .4°. de la mesura de la spera succintamente fosse ditto abastança: non dimeno me par qui excelso Duca particularmente dire de alquanti corpi essentiali in ditta spera locabili de li quali unangolo toccando subito tutti toccano: e principalmente lo fo per

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NALLI (II, 1848, 791, n. 2); Società di amatori delle arti belle (IV, 1848, 14, n. 4); G. MILANESI (III, 1878, 487, n. 2). Also: separate ed. of the *vita* of PIERO, with notes by F. GHERARDI DRAGOMANNI (Florence 1835), p. 19, n. 1; G. BOSSI, *Del Cenacolo di Leonardo da Vinci* (Milan 1810), p. 10; G. TIRABOSCHI, *Storia della letteratura italiana,* VII, part 2 (Milan 1824), 751; L. PUNGILEONI, in *Giornale arcadico di scienze ed arti,* LXII (1835), 214ff; LXIV (1837), 186ff; G. GAYE, in *Kunstblatt,* XVII (1836), no. 69, 284-7; E. HARZEN, in *Naumann's Archiv für zeichende Künste,* II (1856), 237; H. JANITSCHEK, in *Kunstchronik,* XIII (1879), 670-74. Unknown to JORDAN, the manuscript of the *Libellus* had already been identified by J. DENNISTOUN, around 1850, in the Vatican library. See his *Memoirs of the Dukes of Urbino,* 2nd ed. (London 1909), II, 205-06. The *Trattato d'abaco* may well have been the manuscript by Piero that disappeared from the library of the Franceschi Marini family, Piero's descendants, around 1835 and that found its way into G. Libri's possession (above, Chapter II, p. 22). See VASARI, *Cinque vite annotate,* ed. G. Mancini (Florence 1917), p. 210, n. 2. A close reading of the commentaries to Vasari's *vita* of Piero shows that around 1832 (MASELLI, p. 297, n. 2) there were manuscripts by Piero in the possession of the Franceschi Marini in Sansepolcro. By 1848, however (Società di amatori delle arti belle, IV, 13 n. 2), they were no longer there.

[4] *Summa arithmetica,* Part II, fols. 68v-73v.

> la notitia de li .5. regulari di quali Euclide a pieno nelli ultimi soi libri scientificamente tratta. Di che me pare non inutile a ponere de loro certi casi acio li pratici vulgari ancora essi qualche dolceçça di loro dimensioni sentino. Di questi fra philosophy si fa gran discussioni. E maxime se bene el thymeo del Divin philosopho Platone (secondo lo Aurelio doctor sancto Augustin) con diligentia fatende. Dove de universi natura diffusamente parlando spesso a suo proposito li induci. Attribuendolo lor forme separatamente ali .5. corpi semplici: cioè. Terra. Aqua. Aeri. Fuoco. E Cielo... Siche di questi le sequenti petitioni inquiriranno el loro modo operativo abastança pratica insegnaremo commo apresto se intendera [5].

The « *corpora regularia et ordinaria* » of the title are the five regular bodies. Succinctly Pacioli defines them as just those bodies that, if placed in a sphere and if one of the vertices touches the surface of the sphere, then all the vertices will do so. He also states that they are fully treated by Euclid in the last books of the *Elements,* and he recalls the Platonic significance of the five regular bodies as attributes of earth, water, air, fire, and the heavens. The fifty-six exercises that follow Pacioli's introduction treat, for the most part, tetrahedra, cubes, irregular eight and fourteen-faced solids, dodecahedra, icosahedra, octahedra, and the sphere. These solid geometry lessons in Pacioli's « Particularis tractatus » were taken from Piero della Francesca's *Trattato d'abaco,* folios 105r through 120r, and they are the stereometrical exercises we discussed above in Chapter II. Over half of the problems were reproduced with no change from Piero's work, some were made more concise, others considerably shortened. A few were lengthened for further clarification and a few totally reworked [6]. All in all, what we find at the end of Pacioli's *Summa*

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[5] Ibid., fol. 68v.

[6] Twenty-seven of PACIOLI'S exercises are reproduced from the *Trattato d'abaco* almost word for word. They are nos. 3-5, 11, 14, 16, 19, 20, 25, 27-30, 32, 33, 35, 37-44, 46, 54, 55. Ten were made more concise by

arithmetica represents Piero's preliminary work towards the *Libellus de quinque corporibus regularibus,* the solid geometry lessons of his *Trattato d'abaco* [7].

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shortening sentences, or minor changes in wording — nos, 6, 9, 10, 13, 15, 17, 21, 26, 34, 48. Six were considerably abbreviated — nos. 2, 7, 8, 22, 52, 56; three lengthened for clarification — nos. 12, 45, 47; four partially reworked — 18, 51, 53, and the second problem numbered 53. Only five were completely revised, nos. 23, 24, 36, 49, 50. See Table I for the corresponding exercise numbers in the *Summa arithmetica* and the *Trattato d'abaco.*

[7] Exercise 31 in the *Summa* has no corresponding exercise in the *Abaco.* It does not, however, represent an addition by Pacioli because it appears later in PIERO's *Libellus* (II, 37). This circumstance suggests that PACIOLI had before him, not the Laurenziana ms. of the *Abaco,* but rather a faithful version of it. On the other hand exercise no. 4 on fol. 114r of PIERO's *Abaco* does not appear at all in PACIOLI's *Summa.* Nonetheless, a knowledge of its solution was necessary to follow the solution to *Summa* no. 35. Its omission, by Pacioli or the printer, was undoubtedly unintentional.

Inasmuch as Pacioli worked from a manuscript version of the Laurenziana *Abaco,* not the Laurenziana *Abaco* (Ashb. 280) itself, does the « Particularis tractatus » of PACIOLI's *Summa arithmetica* reproduce this lost manuscript of PIERO's *Abaco* faithfully, as was the case with PIERO's *Libellus* printed in PACIOLI's *Divina proportione?* In other words, who was responsible for the divergences of the « Particularis tractatus » from the *Abaco,* as we have it in the Laurenziana version? Piero, in an intermediate manuscript devoted to the regular bodies, or Pacioli himself? Pacioli's responsibility for the changes is indicated by a close comparison of the 38 problems common to the *Summa,* the *Abaco,* and the *Libellus* (compare Tables I and II). Were Piero responsible for the variations introduced in the regular bodies treatise of the *Summa,* some movement towards the later *Libellus* should be discernible in these modifications. This is not the case. The exercises in the *Summa* that differ most markedly from their predecessors in the *Abaco* are nos. 23, 24, 36, 49, and 50. These problems relate no more closely to the corresponding exercises in the *Libellus* than they do to those in the *Abaco.* Thus the authorship of the variations between the *Abaco* and the « Particularis tractatus » must be assigned principally to Luca Pacioli. This conclusion is confirmed by the character of some of the minor variations in wording, clarity, and numerical simplification, which may also be assigned to Pacioli. Furthermore, in some of the variant passages Pacioli actually refers back to earlier parts of the *Summa* proper, for instance, no. 23: « *te mostrai ala terça di [stintion] nel .6.° capitolo,* » and in the same problem: « *commo to mostro al suo luogo nella arithmetica.* »

A representative example of Pacioli's use of Piero's work is exercise 29 of the *Summa,* where it is demonstrated how to find the diameter of a sphere into which an icosahedron, whose area is 200 braccia, has been inscribed. We read in the *Summa*:

> Eglie un .20. base triangulari equilatero la cui superficie e braccia .200. domando del diametro de la spera dove sia descripto. Tu hai per la precedente che el .20. base che la sua superficie sia .200. che il suo lato e .R R. 533 1/3. e per la 2.a del .20. base hai che illato che e .4. de diametro .40. per. R.320. E perche tu hai illato che .R R. reca .4.a. .R R. fa .256. e reca .40. a .R. fa .1600. e poi reca a .R.320. fa .102400. e hai .1600. per .R.102400. Ora di cosi se .256. de lato da de diametro .1600. per .R.102400. che dara .533 1/3. Montiplica 533 1/3 via .1600. fa .853333 1/3. parti per .256. nevene .3333 1/3. ora reca a .R.533 1/3. fa 284444 4/9. montiplicalo con .102400. fara .29127311111 1/9. el quale parti per .256. recato a .R. fa .65536. nevene .429188 26068/32489. e hai 3333 1/3 per .R. 429188 26068/32489 adonca di chel diametro de la spera dove e discripto ditto corpo base .20. triangolari che la sua superficia sia .200. sia la .R. de la summa che fa la .R R. 429188 26068/32489. posta sopra la .R. 3333 1/3 ecc.

This is clearly taken from the *Trattato d'abaco,* fol. 113r (3).

> Egl'è uno 20-baxe triangulare equilatero, che la superficie sua è 200 bracci. Domando del diametro de la spera dov'è descricto. Ai, per la precedente, che il 20-baxe che la superficie sua è 200, che il suo lato è radici de la radici de 533 1/3. Et per la seconda de' 20-baxe ài che il lato ch'è 4 dà de diametro 40 più radici de 320. Et perché tu ài il lato ch'è radici de radici, reca 4 a radici de radici fa 256, et reca 40 a radici fa 1600 et reca 320 a radici fa 102400, et ài 1600 più radici 102400. Hora di' così: se 256 de lato dà de diametro 1600 più radici de 102400, che darà 533 1/3? Montiplica 533 1/3 via 1600 fa 853333 1/3, partilo per 256 ne vene 3333 1/3. Hora reca a radici 533 1/3 fa 284444 4/9, montiplicalo con 102400 fa 29127311111 1/9, il quale parti per 256 recato a radici, ch'è

> 65536, ne vene 429188 26068/32489. Et ài 3333 1/33 più radici de 429188 26068/32489. Adunqua di' che il diametro de la spera, dove è descricto il corpo de 20 baxe triangulare che la sua superficie è 200, sia radici de la somma che fa la radici de la radici de 429188 26068/32489 posta sopra la radici de 3333 1/3.

Absolutely conclusive instances of Pacioli's use of Piero's work in the *Summa* are found in exercises 19 and 20. These represent two exceptions in Piero's *Abaco,* the two problems in measurement which do not concern abstract polygons and polyhedra but rather everyday objects. They treat the determination of the « *quadratura* » of a barrel and that of a statue. Problems in measuring barrels are frequently encountered in *abbaco* handbooks, but the measurement of a statue is apparently unusual. (In the method of solution, however, it makes little difference what kind of a solid one measures.) Here is Pacioli's solution; the changes he introduces in Piero's text are italicized.

> Per che ale volte acade a mesurare corpi inregolari quali non se posano mesurare per linee, commo statue de marmo o de metallo cioe figure de animali rationali o inrationali dico che tenga questo modo a quadrarli *apresso quello che di sopra sucintamente dissi in la domanda* .43. *verbigratias.* Metiamo che tu voglia sapere questo e quadrata una statua de homo nuda che sia .3. braccia de longheçça proportionata. Dico che tu faccia un vaso de legno longo braccia .3 1/4. e largo .1 1/2. e alto .1. el quale sia quadro: cioe con angoli retti e sia bene stagno siche laqua non desca ponto: poi lo metti bene piano alivello e metti dentro tantaqua che agionga a un terço alorlo di sopra. Poi fa un segno dove agiongni laqua e mettivi dentro la statua che tu voi mesurare e vedi quanto e cresciuta laqua e fa unaltro segno a sommo laqua ritto a quello de prima. Poi mesura dal primo segno al secondo e vedi quanto eglie: mettiamo che sia 1/4 ora montiplica la longheçça del vaso che .3 1/4. con la largheçça che .1 1/2. fara 4 7/8. el quale montiplicata per 1/4 che creve laqua fa .1 7/32. e tanto e quadrata

> ditta statua: e così obsua amesurare tali corpi *se fosse ben un par de buoi con un carro de fieno ecc.*

Pacioli referred to his inclusion of the « Particularis tractatus circa corpora regularia et ordinaria » in the *Summa arithmetica* in Chapter 69 of his *Divina proportione,* which dates from 1497 [8]. The measurement of the five regular bodies, he wrote, would not be treated in the *Divina proportione* because he had already composed a treatise on their measurement, dedicated to Guidobaldo da Montefeltro which was easy enough to come by.

> Segue a doverse dire dela dimensione deli corpi regulari e de loro dependenti. Onde de dicti regulari non mi curo altramente qui extenderme per haverne gia composto particular tractato alo illustrissimo affine da vostra Ducale celsitudine Guido Ubaldo Duca de Urbino nella nostra opera a Sua Signoria dicata. e al lectore facile a quella sia el recorso per essere ala comune utilita pervenuta commo dinanze fo detto. E in questa vostra inclita cita asai se ne trovano. La cui mesura tanto e piu speculativa quanto più degli altri corpi sonno quelli più excellenti e perfecti [9].

Pacioli writes « *per haverne gia composto particular tractato* », and his introduction to the « Particularis tractatus » itself offers no indication of another author. Thus there was no reason to believe Pacioli was not its author. In the *Summa arithmetica,* then, there is a further substantiation of Vasari's contention that Fra Luca Pacioli published the works of « quel gran vecchio » under his own name. One further conclusion may be drawn concerning Pacioli's 1509 publication of the *Libellus de quinque corporibus regularibus.* The reader of the

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[8] L. Pacioli, *Divina proportione* (Venice 1509), colophon on fol. 23r.
[9] Ibid., fol. 21v.

manuscript version of Pacioli's *Divina proportione* (1497) who wanted to learn about the measurement of regular bodies is referred to the earlier treatise on that subject published in Pacioli's *Summa de arithmetica, geometria, proportioni et proportionalità* rather than to the *Libellus,* immediately at hand in the *Divina proportione*. Thus it must have been after the completion of his manuscript and toward the time of publication of the book that Pacioli decided to translate and publish the *Libellus,* which he appended to the printed edition of the *Divina proportione* of 1509.

Appendix II

Piero's Manuscripts and Pacioli's Adaptations: Textual Questions

As we have seen, Luca Pacioli's work on the five regular bodies was largely derived from Piero della Francesca's. Both Fra Luca's « Particularis tractatus circa corpora regularia et ordinaria, » contained in his *Summa de arithmetica, geometria, proportioni et proportionalità* of 1494, and his « Libellus in tres partiales tractatus divisus quinque corporum regularium et dependentium », printed in his *Divina proportione* of 1509, represent little more than versions of two of Piero's treatises. Piero's *Trattato d'abaco* (Bibl. Laurenziana, Ashb. 280) provides the basis for the regular body treatise Pacioli printed in his *Summa arithmetica,* and Piero's *Libellus de quinque corporibus regularibus* (Vat.Urb.lat. 632), the basis for the regular body treatise Pacioli subsequently published in his *Divina proportione.*

This double set of parallel circumstances raises several questions whose answers offer some insight into the character of Piero's original manuscripts.

Did Piero compose the *Libellus* in Latin, as sometimes suggested [1], or did he write it in Italian and have it translated to

[1] For instance, L. Olschki, *Geschichte der neusprachlichen wissenschaftlichen Literatur* (1919; rpt. Vaduz 1965), I, 216, n. 2.

present to the Duke of Urbino? If Piero's Latin *Libellus* represents, then, the work of a translator, is Pacioli's Italian version of it, contained in the *Divina proportione,* a successive translation into Italian, or is it simply the publication of Piero's original Italian manuscript? And finally, to what extent, if at all, did Pacioli revise Piero's material before he published it?

The answers to these questions arise from a close comparison of the four texts in question. To anticipate conclusions, it seems clear that Piero wrote the *Libellus* in Italian, and that it was then translated into humanist Latin, probably by Matteo da Borgo [2]. In turn, Pacioli's version of it, published in 1509, represents a re-translation into Italian, not Piero's original text. In addition, although much of Piero's *Libellus* derives from his *Trattato d'abaco,* there are significant differences between the two works. At the least they represent separate phases in the development of his mathematics. Possibly Pacioli perceived this development. But, in his borrowings, Pacioli made no attempt to reconcile the variances the two manuscripts presented him. Although minimally he had to occupy himself with the translation of one of the manuscripts, Pacioli seems to have been more interested in expanding the bulk of his literary oeuvre through a policy of editorial aggrandizement.

The demonstration of these conclusions is dependent, in each case, upon detailed comparisons of texts and language, and sometimes of mathematical procedure. Some trusting readers may wish to spare themselves this material.

The geometrical exercises of Piero's Latin *Libellus* are rendered in Italian, with only slight changes in numeration, at the end of Pacioli's *Divina proportione.* A comparison of

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[2] See above, Chapter III, n. 8.

the language used in the two treatises is instructive. Many Latin words or expressions that render fragmentary sentences whole, or sharpen syntax, or permit variety in diction, simply do not have correspondences in normal late Quattrocento written Italian. Thus, where Pacioli's text repeats and repeats the same limited fund of words, the *Libellus,* written in Latin for the Urbino court, can avail itself of a wealth of expressions to formulate the same ideas. Consequently its literary effect is of a much higher order. Some examples illustrate this point. Pacioli's *resta* is expressed in Latin as *superit,* as *faciet reliquum,* as *relinquitur,* and as *residuum est. Leva* is rendered: *aufert, removet, tollit;* similarly, *trallo*: *deducas, subtrahe, demas; piglia*: *cape, capitur, sume, sumatur.* The ubiquitous *fa* of Pacioli's text is represented in a variety of expressions: *reddentur, reddet, reddit, reddunt, agendum est, dabit, efficit, efficietur, habebis, habemus, facit, fient, fuit, fit, fuerunt, eveniet* and *existat.* Furthermore, Pacioli abbreviates in Italian many expressions that are given completely in Latin.

Since the problems in Pacioli's « Particularis tractatus », in his *Summa,* derive from Piero's *Abaco,* and since the *Abaco* was, in turn, a preliminary step to Piero's *Libellus,* the correspondences among these works will bear examination. We may compare the *Divina proportione* translation of Piero's *Libellus* both to the *Trattato d'abaco* and to the Latin *Libellus* to determine to which it is more closely related. If the *Divina proportione* tract appears close to the *Abaco* text in similar problems, then it perhaps represents Piero's original Italian text. On the other hand, if it is textually closer to the Latin *Libellus,* it may be considered a translation by Pacioli into Italian.

The first comparison presents the case of a regular tetrahedron whose side is $\sqrt{24}$ and whose axis is 4. What then is the length of a line drawn from an angle of the tetrahedron to the center of the axis. The three following textual examples

(I, 1-3) show the *Divina proportione* « Libellus » to be closer to the Vatican *Libellus* in phrasing (and also in diction and method of mathematical solution) than to the *Trattato d'abaco*. This points to the conclusion that Pacioli's publication represents a translation from the Latin rather than the original Italian text of Piero.

I, 1. *Libellus de quinque corporibus,* II, 6:

> Est quadribasis triangularis aequilatera cuius latus est radix 24, et axis est 4. Quantitas igitur quae a centro est ad quemlibet angulorum reperienda est.
> Habes quadribasim ABCD, cuius quodlibet latus est radix 24: et axis AE est 4: et centrum F est in axe. Et quia est proportio ab AF ad AE, quae est a 3 ad 4, quae est proportio sexquitertia, erit AF 3/4 ipsius AE, qui est 4. Itaque AF est 3. Probatio. Dictum fuit quod unum ex lateribus est 24 et AE 3. Igitur FE est 1, quia AE est 4. Aufer AF, quae est 3: relinquitur 1 FE. Et axis cadit super E, quae est 2/3 catheti BG. Et E est centrum basis BCD, et BG per eam quae praecessit est radix 18. Cape 2/3, fiet radix 8. Trahe lineam BF, per penultimam primi Euclidis, potest quantum duae lineae BE, EF; et BF est 3, et aequat AF, sicut per primam huius probatum fuit. Et BF potest 9, et EF potest 1, qui ablatus a 9 relinquetur 8, quae est vis BE, quae addita ad vim EF, quae est 1, efficit 9, et radix 9 est BF, quae est 3, et AF 3, CF 3, DF 3, nam omnes ortum habent a centro F, et in circumferentia terminantur.

I, 2. *Divina proportione,* II, 5:

> Eglie uno quatro base triangulare equilatero che il suo lato e R .24. et laxis e .4. la quantita che dal centro a ciascuno angulo se vole trovare.
> Tu ai il quatro base .a.b.c.d. che ciascuno suo lato e R .24. Et laxis .a.e. e .4. et il centro .f. e nel axis et per che quella proportione e da .a.f. ad .a.e. che da .3. ad .4. che proportione sexquitertia sita .a.f. trequarti de .a.e. che .4. adunqua .a.f. e .3. ala prova e se dicto che uno di lati e R .24. et .a.f. 3. dunqua

.f.e. e .1. perche .a.e. e .4. tranne .a.f. che .3. resta .1. f.e. et laxis cade sopra .e. che li doi terçi del cateto .b.g. et .e. e centro dela basa .b.c.d. et .b.g. per la precedente e R .18. pigliane .2/3. sia R .8. tira la linea .b.f. per la penultima del primo de Euclide po quanto le doi linee .b.e. et .e.f. et .b.f. e .3. et e equale ad .a.f. comme per la prima de questo fu provato et .b.f. po .9. et .e.f. po .1. trallo de .9. resta .8. che la posança de .b.e. che gionta con la posança de .e.f. che .1. fa .9. et la R .9. e .b.f. che .3. et .a.f. 3, .c.f. 3. d.f.3. per che tucte se puntano dal centro .f. e terminano nela circumferentia.

I, 3. *Trattato d'abaco,* fol. 105v (1):

Egl'è una figura corporea che à quatro base triangulare de anguli e lati equali, che ciascuno lato è radici de 24, et il suo assis è 4. Domando quanto è da ciascuno angulo al centro E.
Tu ài la figura corporea de quatro base triangulare equilatera ABCD, et l'assis suo è AE, et il centro è nell'assis in puncto F. Et perché AE è 4 et AF è sexquitertia ad AE, adunqua AF è 3. La prova. E s'è dicto che uno lato e radici de 24, però pigla la metà de radici de 24 ch'è radici de 6 et detrallo de 24, resta 18, cioè radici de 18 è BH. E l'assis il quale cade in su li doi terçi del diametro BH, ch'è radici de 18, et li 2/3 è radici de 8 che in sé montiplicato fa 8; et e' ss'è dicto che AF è 3, dunqua FE è 1. Perché AE è 4, tranne AF ch'è 3 resta FE, 1, che in sé multiplicato fa 1, gionto con 8 fa 9. Adunqua BF, CF, DF è ciascuno radici de 9, ch'è 3.

A second set of comparisons (II, 1-3) reinforces the impression created by the first. In this instance Piero treats the determination of the sides of a regular tetrahedron placed in a sphere whose axis is 7. Again, Pacioli's example in the *Divina proportione* appears as a translation from the Latin.

II, 1. *Libellus de quinque corporibus regularibus,* II, 3:

Quadribasis triangularis aequilateri et a spera contenti cuius diameter est 7, latus reperiendum.

Per eam quae praecessit habes quod ea est proportio ab axe ad latus, quae est a latere ad diametrum sperae eum continentis. Habesque quod posse axis ad posse sui lateris est sexquialterum. Et similiter vis lateris ad vim diametri eadem proportione tenetur. Nunc habes diametrum, qui est 7, et posse suum est 49. Igitur posse diametri sperae est posse lateris quadribasis, et sicut 3 ad 2. Itaque dic: si 3 esset 49, quod erit 2? Multiplices bis 49, habebis 98: partire per 3, eveniunt 32 2/3, cuius radix est latus quadribasis a spera contenti, et cuius diameter est 7.

II, 2. *Divina proportione,* II, 2:

Del quatro base triangulare equilatero contenuto dala spera che il suo diametro e .7. de lato suo investigare.
Per la precedente ai che glie quella proportione dalaxis allato che e dal lato al diametro dela spera chel contene et ai che la posança delaxis ala posança del suo lato e sexquialtera et cosi quella dallato e al diametro hora tu ai il diametro che .7. e la sua posança e .49. adunqua la posança del diametro dela spera e la posança del lato del quatro base si commo .3. ad .2. pero di se .3. fusse .49. che seria 2. multiplica .2. via .49. fa .98. parti per 3 nevene .32 2/3. e la R .32 2/3. e ilato del quatro base contenuto dala spera che il suo diametro e 7.

II, 3. *Trattato d'abaco,* fol. 107r (1):

Egl'è uno corpo sperico che il suo diametro è 7; vogloci mectere dentro una figura de 4 base triangulare equilatera, che gl'anguli contighino la circumferentia. Domando de' suoi lati.
Devidi prima il diametro suo, ch'è 7, in tre parti equali che sirà ciascuna 2 1/3, montiplica 2 1/3 co' lo resto, ch'è 4 2/3, fa 10 8/9, il quale adoppia conmo radici fa radici da 43 5/9. De la quale radici fa 4 parti equali, cioè reca 4 a radici fa 16, parti 43 5/9 per 16; reca le parti a noni, arai 392 a partire per 144, ne vene 2 52/72, trallo de 43 5/9 resta radici de 24 1/2 che è diametro de la basa del corpo triangulare. Et perché egl'è sexquitertia ad il lato de la basa, giogni 1/3 de 24 1/2, ch'è 8 1/16, ad 24 1/2 fa 32 2/3; et la radici de 32 2/3 sirà per il lato il dicto 4-base triangulare che intra nel corpo sperico che il suo diametro è 7 bracci.

That Pacioli did not compare Piero's two treatises but adopted each independently is clear if the corresponding problems in the *Summa* and the *Divina proportione* and the *Libellus* are compared, because of the textual relations that emerge. The correspondences between the *Divina proportione* and the *Libellus* are far closer than those between the *Summa* and the *Libellus*. Two sets of examples illustrate this situation: III, 1-3 and IV, 1-3.

III, 1. *Libellus de quinque corporibus regularibus,* II, 18:

> Sit cubus ABCDEFGH, habens latera quaeque ulnarum 4: quaerendum est quot ulnarum erit quadratus?
> Dictum fuit in principio quadratorum, quod eorum quadratura a suis lateribus habentur, si suum latus reduceretur ad cubum. Igitur multiplica suum latus in se, quod est 4, fiet 16, et quater 16 reddit 64. Dices ergo quod cubus ABCDEFGH, cuius latus sit 4, quadratus erit 64.

III, 2. *Divina proportione,* II, 17:

> E se il cubo a.b.c.d.e.f.g.h. e per ciascuno lato .4. quanto sira quadrato se vole cercare.
> Fu dicto nel principio de quadrati che la sua quadratura favia dai suoi lati cioe recando il suo lato a cubo poi multiplica il suo lato che .4. in se fa .16. et .4. via .16. fa .64. adunqua dirai che il cubo .a.b.c.d.e.f.g.h. che al suo lato .4. sia quadrato .64.

III, 3. *Summa arithmetica,* 6:

> Eglie un cubo .abcd.efgh. che per ciascun lato e .4. braccia. dimando quanti braccia sira quadro. Moltiplica 4 in se fa .16. e poi .16. via .4. fa .64. e tanto e quadro facta.

The relationships that already begin to be evident from the

comparison of these three versions of a single problem can be traced in greater detail in the three redactions of the following problem (IV, 1-3).

IV, 1. *Libellus de quinque corporibus regularibus,* II, 15:

> Sit latus cubi quadrilateri 4: quot erit diameter sperae, quae eum circumscribit investigandum est.
> Dico quod proportio possibilitatis diametri sperae ad eam quae est lateris cubi in illa descripta est tripla: hoc est sicut 3 ad 1. Ideo multiplica latus cubi in se, quod est 4, fiet 16. Dic praeterea. Si 1 esset 16, quot essent 3? Multiplica per 16, fiet 48, quem partire per 1, evenient 48: et 48 est vis diametri sperae quae cubum continet. Igitur diameter sperae est radix 48. Sed ut clarius intelligatur: habes cubum ABCDEFGH, extendas lineam AD, quae per penultimam primi Euclidis, potest quantum hae duae lineae AB et BD, quarum quaelibet est 4, et in se multiplicatae cum multiplicatione sua efficiunt 32. Itaque vis AD est 32. Et si trahas AH, eadem ratione, potest quantum duae lineae AD et DH, quae continent angulum D, qui rectus est. Et DH est 4, quae potest 16, et AD potest 32, qui additus ipso 16 efficit 48, quae est vis AH, quae linea transit per centrum cubi et sperae. Et angulus A, et angulus H contingunt circumferentiam sperae. Igitur AH est diameter sperae, et sua vis est 48, et circumscribit cubum, cuius vis lateris est 16, quae est 1/3 possibilitatis diametri.

IV, 2. *Divina proportione,* II, 14:

> Se illato del cubo equilatero e .4. che sira il diametro dela spera che il circumscrive investigare.
> Dico che la proportione dela posança del diametro dela spera aquella dellato del cubo in quella descricto e tripla cioe como .3. ad uno pero multiplica illato del cubo che .4. in se fa .16. hora di se uno fusse .16. ch saria .3. multiplica .3. via .16. fa .48. il quale parti per uno ne ven. 48. et .48. ela posança del diametro dela spera che contene il cubo adunqua il diametro delaspera e R de .48. E per che meglio lo intenda tuai il cubo .a.b.c.d.e.f.g.h. tira la linea .a.d. laquale per la penultima del primo de Euclide po quanto le do linee .a.b. et .b.d. che ciascuna .4. che multiplicata ciascuna inse egionte insieme le multi-

plicationi fano .32. dunqua la posança de .a.d. e .32. et se tu tiri .a.h. per quella medesima ragione po quanto le do linee .a.d. et .d.h. che contengano langulo .d. che recto et .d.h. e .4. che po .16. et .a.d. po .32. che gionto con .16. fa .48. che la posança de .a.h. la quale linea passa per lo centro del cubo e de la spera et langulo .a. e langulo .h. contingano la circumferentia dela spera adunqua .a.h. e diametro dela spera e la posança sua e .48. et circumscrive il cubo che la posança del suo lato e .16. ch .1/3. dela posança del diametro.

IV, 3. *Summa arithmetica,* 7:

Eglie un cubo ch' per faccia .4. dimando che sia la diagonale interiore passante per lo centro K cioe laxis. Opera e troverai che sira la diagonale .ah. R .48. facta.

If the corresponding problems in Piero's *Trattato d'abaco* (III, 4 and IV, 4) are joined to the two preceding sets of comparisons, it is confirmed that Pacioli did not attempt to reconcile the discrepancies that exist between Piero's texts. Instead, he translated or directly copied, often abbreviating the originals.

III, 4. *Trattato d'abaco,* fol. 106r (1):

Adunqua egl'è uno cubo ABCD.EFGH che è, per ciascuno lato, 4 bracci. Domando quanti bracci sirà quadrato. Tu di' ch'egl'è 4 bracci per lato; dunqua montiplica 4 via 4 fa 16, et 4 via 16 fa 64: però dirai ch'egl'è quadrato 64 bracci.

IV, 4. *Trattato d'abaco,* fol. 106r (3):

Egl'è uno cubo ch'è 4 per lato. Domando de la linea AH, passante per lo cintro K, de l'angulo A a l'angulo H.
Trova prima la linea diagonale AD che ài che è radici de 32, conmo per al precedente; poi montiplica DH, ch'è 4, in sè fa 16, giognilo con 32 fa 48 et radici de 48 sirà il diametro AH pasante per lo centro K.

These examples demonstrate that the « Particularis tractatus » in Pacioli's *Summa arithmetica* derives exclusively from Piero's *Trattato d'abaco,* and, further, that the « Libellus » in Pacioli's *Divina proportione* is translated directly, and also exclusively, from Piero's *Libellus* in the Vatican [3].

A look at phrasing and mathematical procedure serves to confirm the above conclusions. Comparing, for instance, the problems concerning the measurement of a statue found in all four treatises, we read as follows — *Summa,* 20: « *Dico che tu faccia un vaso de legno...* »; *Abaco,* fol. 109v (1): « *Dico che tu facci uno vaso de legno...* »; *Divina proportione,* IV, 17: « *Fa uno vaso de legno ho d'altro...* »; *Libellus,* IV, 17: « *Conficias vas ex ligno, aut ex alia re...* ». The phrase, « *aut ex alia re* », found in both the Vatican *Libellus* and in the *Divina proportione* appears neither in the *Trattato d'abaco* nor in the *Summa arithmetica.* « *Conficias* » in the Vatican treatise and in the *Divina proportione* is the less succinct « *Dico che tu facci* » in the *Summa* and *Trattato d'abaco* [4].

This problem concerning the measurement of a statue confirms, as well, that Pacioli did not draw upon both of his sources in order to refine his method of mathematical solution. In

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[3] One may further compare *Summa,* 4: *Tu ai la figura corporea de .4. base; Abaco,* fol. 105v (1): « *Tu ai la figura corporea de quattro base* »; *Libellus de quinque corporibus,* II, 6: « *Habes quadribasim ABCD* »; *Divina proportione,* II, 5: « *Tu ai il quatro base a.b.c.d.* » Or, compare: *Summa,* 6: « *Montiplicata 4 in se fa 16 e poi 16 via 4 fa 64 e tanto e quadro facto;* » and *Abaco,* fol. 106r (1): « *Montiplicata 4 via 4 fa 16, et 4 via 16 fa 64: però dirai ch'egl'è quadrato 64 bracci.* » These last two examples represent the full solutions. The Vatican *Libellus,* however, and following it, the *Divina proportione* « Libellus, » both contain a preface to this solution — *Libellus,* II, 18: « *Dictum fuit in principio quadratorum, quod eorum quadratura a suis lateribus habentur, si suum latus reduceretur ad cubum* »; *Divina proportione,* II, 17: « *Fu dicto nel principio de quadrati ch' la sua quadratura favia da suoi lati cioe recando il suo lato a cubo.* »

[4] The problem is reproduced in full above, from the *Summa arithmetica,* in Appendix I, pp. 104-05.

the Vatican *Libellus,* one marks the height of the water in a vat, puts the statue into the vat, waits for the water to become still, makes a new mark for the higher water level, removes the statue, and measures the distance between the first and second marks. The method is identical in the *Divina proportione.* In the *Trattato d'abaco,* however, one marks the water level, adds the statue, marks the higher water level, and measures the distance between the first and second levels. In the *Summa arithmetica* the same process is demonstrated. The phrase, « *et sinas aquam quietam stare,* » in the Vatican *Libellus,* and « *e lassa reposare l'aqua* », in the *Divina proportione,* occurs in neither the *Trattato d'abaco* nor in the *Summa arithmetica.* Nor is the statue removed from the vat in the latter two treatises [5].

The wording, and phrasing, of identical problems in the *Summa* and in the *Divina proportione* is at times widely variant. The sentence constructions, however, relate closely to the original sources in Piero's manuscripts. While Piero's Italian is simple, at times even elementary, Pacioli's Italian

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[5] We read in the *Summa,* 20: « *Poi fa un segno dove agiogni laqua e mettivi dentro la statua che tu voi mesurare e vedi quanto e cresciuta laqua e fa unaltro segno e sommo laqua ritto a quello de prima. Poi mesura dal primo segno al secondo e vedi quanto eglie;* » and similarly in the *Abaco,* fol. 109v (1): « *Poi poni o fa' uno segno a sommo l'aqua et poi ne mecti dentro la statua che tu voi mesurare e vedi quanto è chrescuta l'aqua e fa' un altro segno a sommo l'aqua, dericto al segno de prima. Poi mesura dal primo segno al secondo e vedi quanto egl'è.* » On the other hand, the *Divina proportione,* IV, 17: « *Poi fa uno segno nel vaso a sommo laqua et poi mecti dentro la statua che tu voi mesurare e lassa reposare l'aqua poi vedi quanto e cresciuta et fa a sommo laqua un altro segno dericto a quello de prima poi trafora la statua et mesura quanto e dal primo segno al secondo,* » and the *Libellus de quinque corporibus,* IV, 17: « *Deinde signa eam partem vasis, quae a summitate aquae tangitur. Postea iustus ponas statuam quam vis metiri, et sinas aquam quietam stare. Deinde videas quantum creverit, et facias aliud signum ad summitatem aquae de directo supra primum, deinde extrahe statuam, et mensura quantum distat secundum signum a primo.* »

in the *Divina proportione* is distinguished by greater concision, clarity, and syntactical rigor. These differences, together with the discrepancies in content, suggest that the *Divina proportione* « Libellus » (despite the simplifications pointed out) is a translation into Italian from a concise humanist Latin text. One perhaps must conclude that Piero's original Italian manuscript for the Vatican *Libellus* had much the same character as the problems in the *Trattato d'abaco,* which are written in a rough, unpolished style, and that the translator was almost entirely responsible for the literary ornaments of the Latin text.

Table I

Derivations of Exercises in the «Summa arithmetica» from the « Trattato d'abaco »

Summa arithmetica ... Part II, fols. 68v-73v problem no.:	Ashburnham 280: *Trattato d'abaco*
2.	fol. 105r (2)
3.	fol. 105r (3)
4.	fol. 105v (1)
5.	fol. 105v (2)
6.	fol. 106r (1)
7.	fol. 106r (2)
8.	fol. 106r (3)
9.	fol. 106r (4)
10.	fol. 106v (1)
11.	fol. 106v (2)
12.	fol. 106v (3)
13.	fol. 107r (1)
14.	fol. 107r (2)
15.	fol. 107r (3)
16.	fol. 107v (1)
17.	fol. 108r (1)
18.	fol. 108v (1)
19.	fol. 109r (1)
20.	fol. 109v (1)
21.	fol. 110r (1)
22.	fol. 110r (2)
23.	fol. 110v (1)
24.	fol. 111r (1)
25.	fol. 112r (1)
26.	fol. 112v (1)

Summa arithmetica ... Part II, fols. 68v-73v problem no.:	Ashburnham 280: *Trattato d'abaco*
27.	fol. 113r (1)
28.	fol. 113r (2)
29.	fol. 113r (3)
30.	fol. 113v (1)
31.	——
32.	fol. 114r (1)
33.	fol. 114r (2)
34.	fol. 114r (3)
—.	fol. 114r (4)
35.	fol. 114v (1)
36.	fol. 115r (1)
37.	fol. 115r (2)
38.	fol. 115v (1)
39.	fol. 115v (2)
40.	fol. 115v (3)
41.	fol. 116r (1)
42.	fol. 116v (1)
43.	fol. 116v (2)
44.	fol. 117r (1)
45.	fol. 117r (2)
46.	fol. 117r (3)
47.	fol. 117r (4)
48.	fol. 117v (1)
49.	fol. 117v (2)
50.	fol. 117v (3)
51.	fol. 118r (1)
52.	fol. 118v (1)
53^1.	fol. 118v (2)
53^2.	fol. 119r (1)
54.	fol. 119r (2)
55.	fol. 119v (1)
56.	fol. 119v (2)

Table II

Derivations of Exercises in the « Libellus de quinque corporibus regularibus » from the « Trattato d'abaco »

Libellus de quinque corporibus regularibus (Vat.Urb.lat. 632)	*Trattato d'abaco* (Bibl. Laurenziana, Ashb. 280)
Part I:	
3.	fol. 80^{v} (1)
4.	fol. 79^{r} (1)
8.	fol. 100^{r} (1)
11.	fol. 101^{r} (2)
14.	fol. 83^{r} (1)
15.	fol. 83^{r} (2)
16.	fol. 83^{v} (1)
17.	fol. 83^{v} (2)
18.	fol. 83^{v} (3)
19.	fol. 84^{r} (1)
20.	fol. 84^{r} (2)
21.	fol. 84^{r} (3)
22.	fol. 102^{r} (1)
23.	fol. 102^{v} (1)
24.	fol. 103^{v} (1)
25.	fol. 103^{r} (2)
28.	fol. 90^{r} (1)
29.	fol. 88^{v} (1)
30.	fol. 88^{v} (2)
34.	fol. 89^{r} (1)
35.	fol. 89^{r} (2)
36.	fol. 90^{r} (1)
37.	fol. 91^{v} (1)
38.	fol. 91^{v} (2)

Libellus de quinque corporibus regularibus (Vat.Urb.lat. 632)	*Trattato d'abaco* (Bibl. Laurenziana, Ashb. 280)
39.	fol. 92r (1)
40.	fol. 92v (1)
42.	fol. 92v (2)
43.	fol. 93r (1)
44.	fol. 93r (2)
45.	fol. 93v (1)
46.	fol. 93v (2)
47.	fol. 93v (3)
49.	fol. 94r (1)
Part II:	
4.	fol. 105r (1)
5.	fol. 105r (3)
6.	fol. 105v (1)
7.	fol. 116r (1)
8.	fol. 115r (2)
9.	fol. 115v (1)
10.	fol. 115v (2)
11.	fol. 115r (1)
12.	fol. 116v (1)
13.	fol. 116v (2)
14.	fol. 117r (1)
15.	fol. 106r (3)
16.	fol. 107r (2)
18.	fol. 106r (1)
21.	fol. 114r (1)
26.	fol. 114v (1)
27.	fol. 110r (2)
28.	fol. 110r (1)
29.	fol. 110v (1)
30.	fol. 111r (1)
31.	fol. 112r (1)
32.	fol. 112v (1)
33.	fol. 113r (1)
34.	fol. 113r (2)
35.	fol. 113r (3)
36.	fol. 113v (1)

Libellus de quinque corporibus regularibus (Vat.Urb.lat. 632)	*Trattato d'abaco* (Bibl. Laurenziana, Ashb. 280)
Part III:	
15.	fol. 106^{v} (2)
16.	fol. 106^{v} (3)
17.	fol. 117^{r} (3)
19.	fol. 117^{v} (1)
20.	fol. 117^{r} (4)
21.	fol. 117^{v} (3)
22.	fol. 117^{v} (2)
23.	fol. 118^{r} (1)
24.	fol. 118^{v} (1)
25.	fol. 118^{v} (2)
27.	fol. 119^{r} (2)
28.	fol. 119^{v} (1)
29.	fol. 119^{v} (2)
Part IV:	
16.	fol. 109^{r} (1)
17.	fol. 109^{v} (1)

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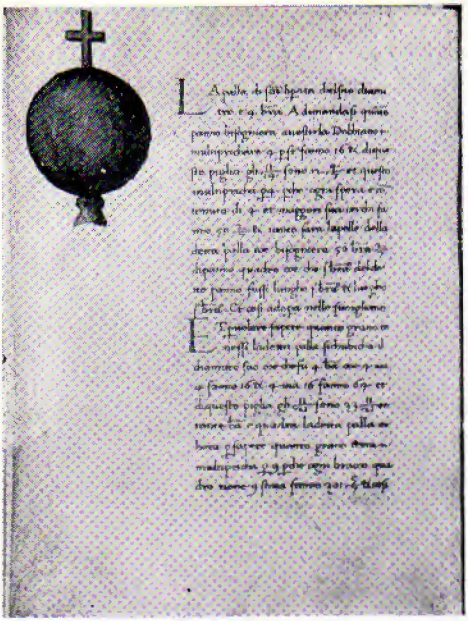
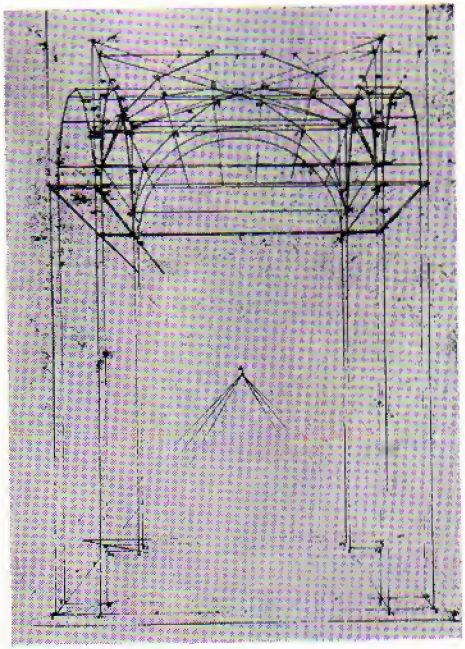
Illustrations

1. Column and barrel. MS. Cod. Acq. e doni 154, fol. 218r.

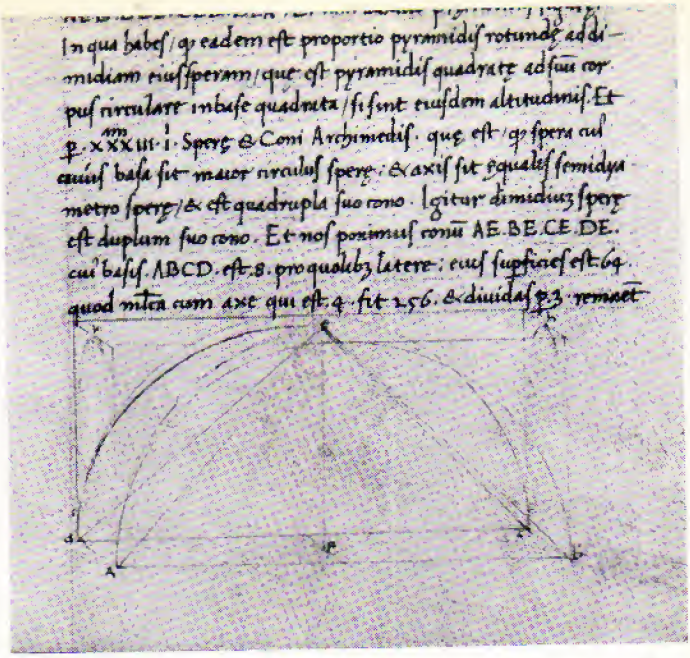
2. Well. MS. Cod. Acq. e doni 154, fol. 220r.



3. «Palla di S. Liperata». MS. Cod. Acq. e doni 154, fol. 219v.

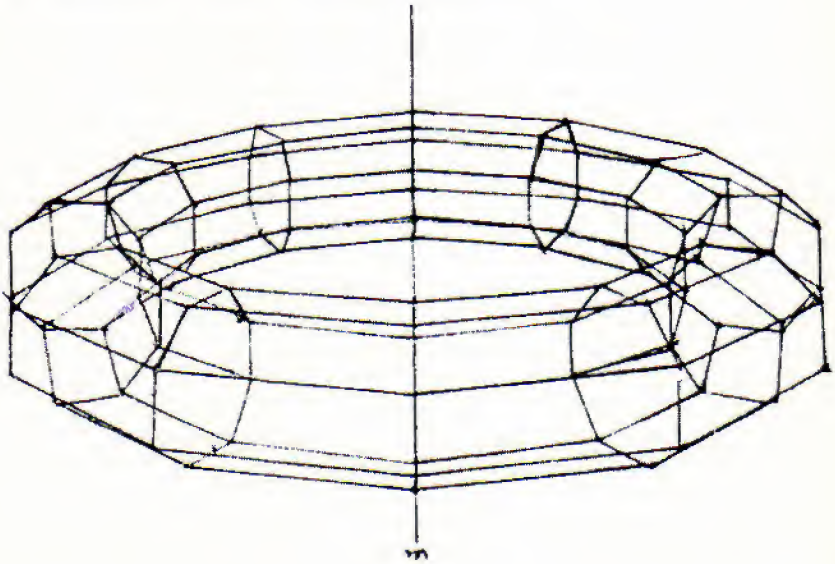


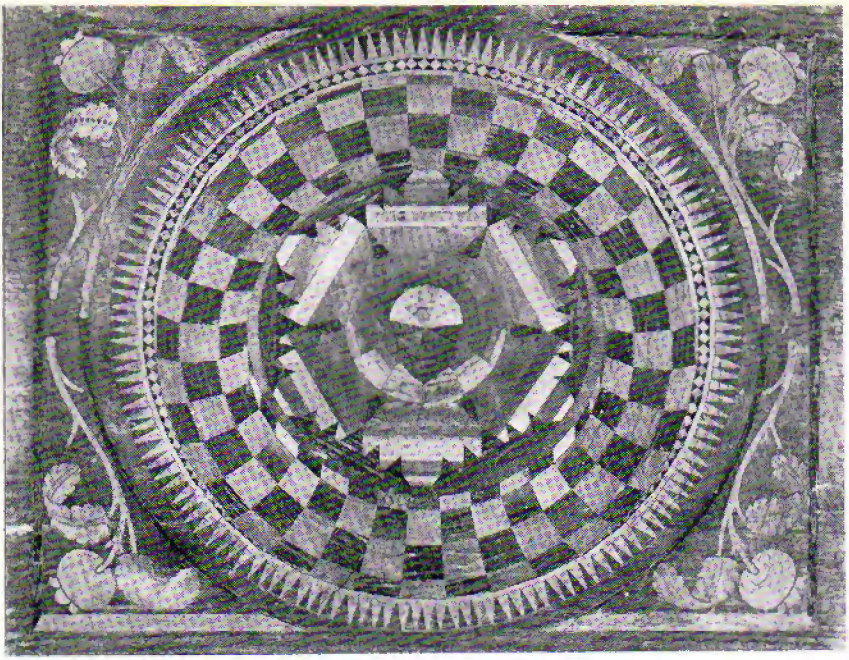
4. « Una volta di crociera ». Piero, *Prospectiva pingendi.*



5. « Volta per modum crucis ». Piero, *Libellus de quinque corporibus*, fol. 61r.

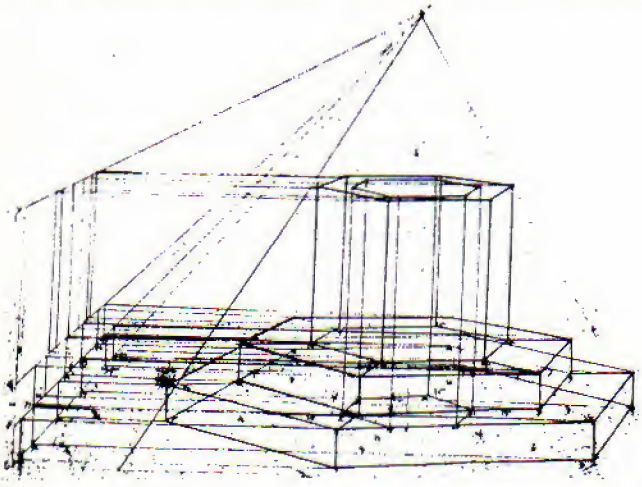
6. *Mazzocchio*. Piero, *Prospectiva pingendi*.

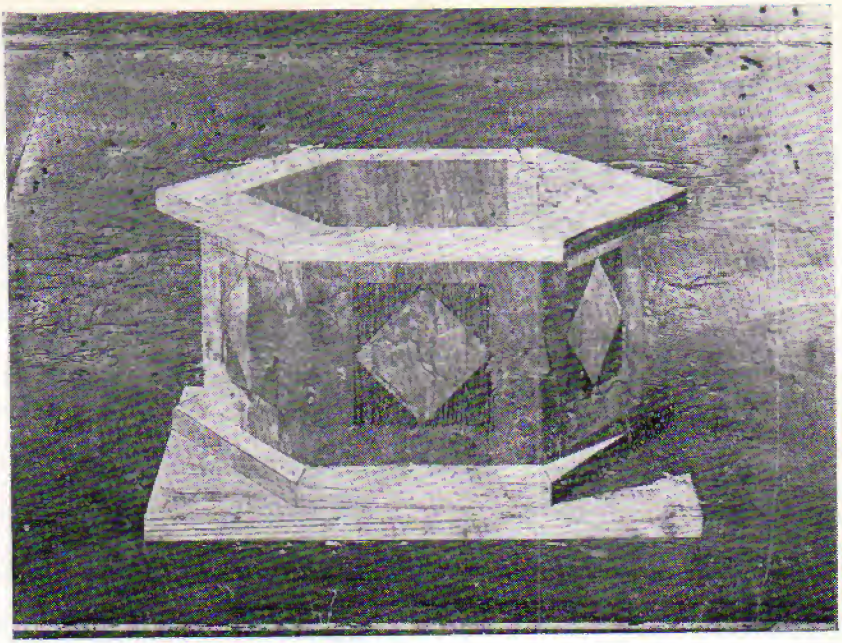




7. *Mazzocchio.* Lorenzo da Lendinara, Duomo, Modena.

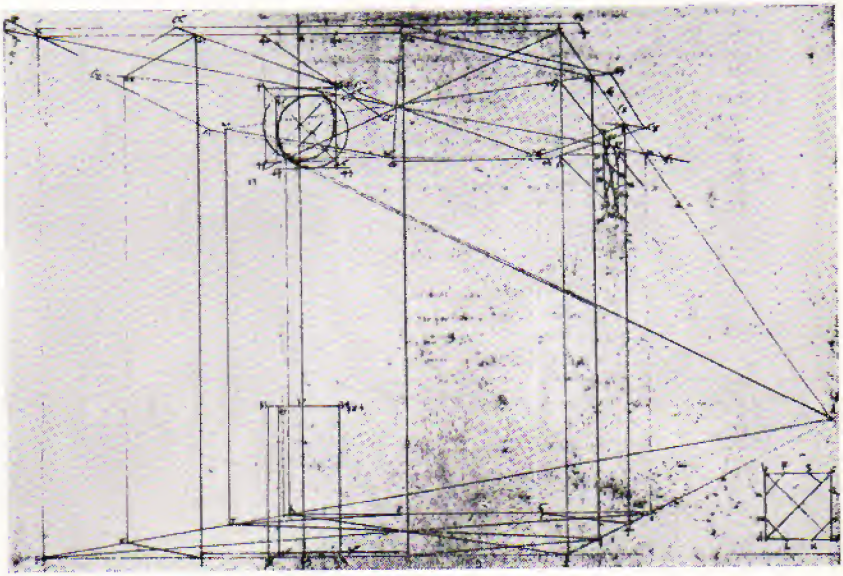
8. «Un pozzo de sei faccie». Piero, *Prospectiva pingendi.*

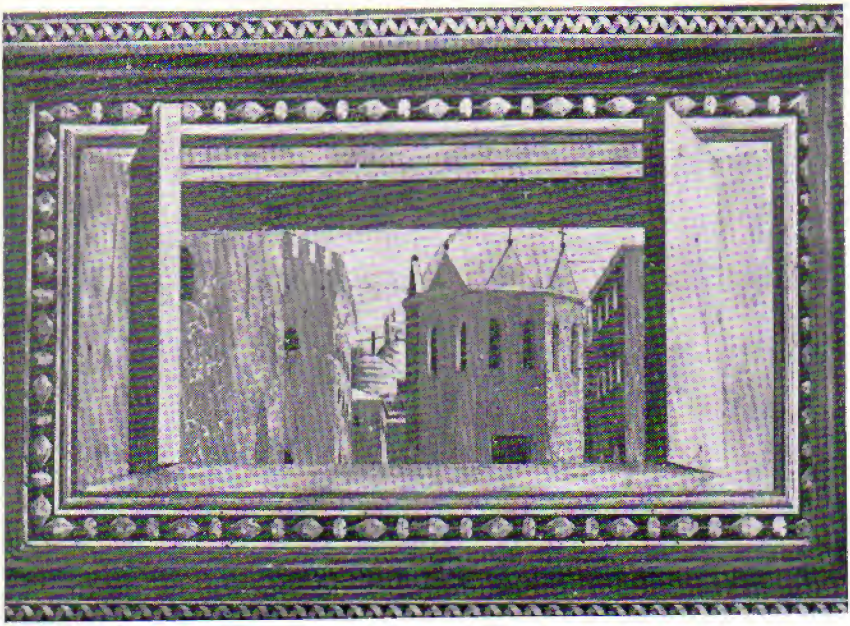




9. Hexagonal well. Lorenzo da Lendinara, Duomo, Modena.

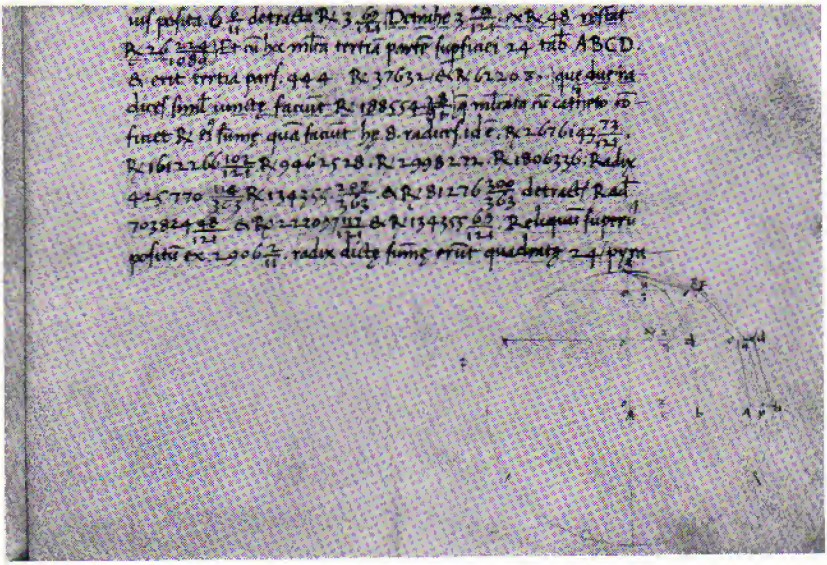
10. «Tempio de octo facce». Piero, *Prospectiva pingendi.*

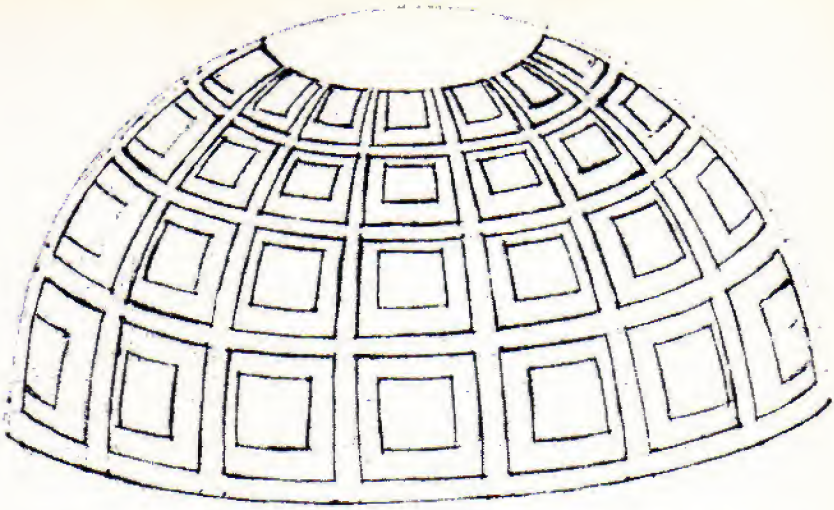




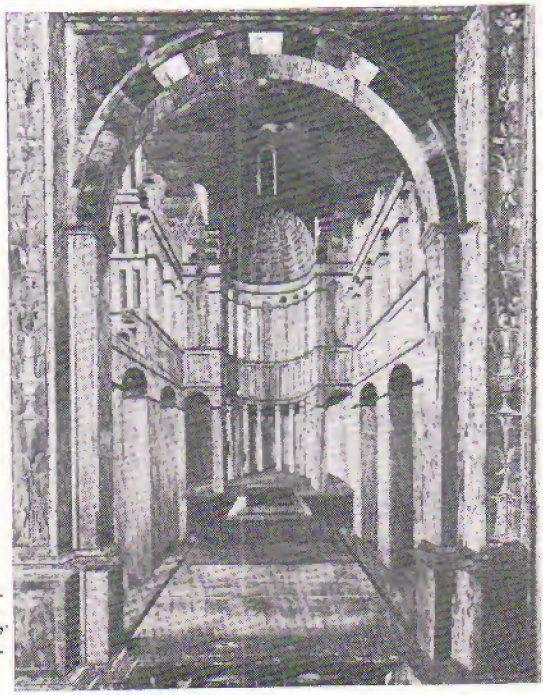
11. Octagonal Baptistery. Lorenzo da Lendinara, Duomo, Modena.

12. «Corpus 72 basium». Piero, *Libellus de quinque corporibus*, fol. 51r.

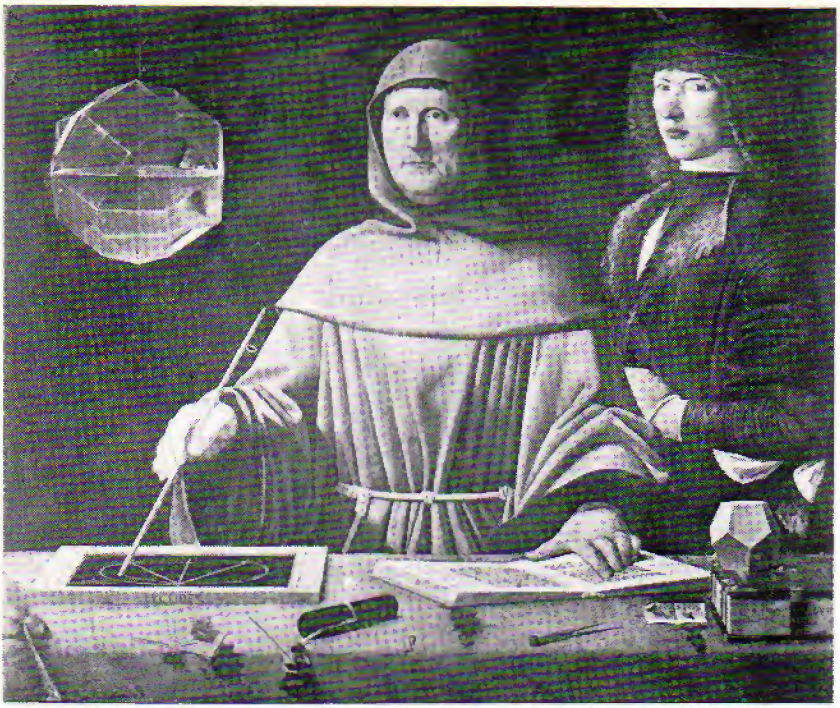




13. «Una cupola... commo uno quarto de una pala». Piero, *Prospectiva pingendi.*



14. Intarsia apse. Giovanni da Verona, Monteoliveto (Siena).

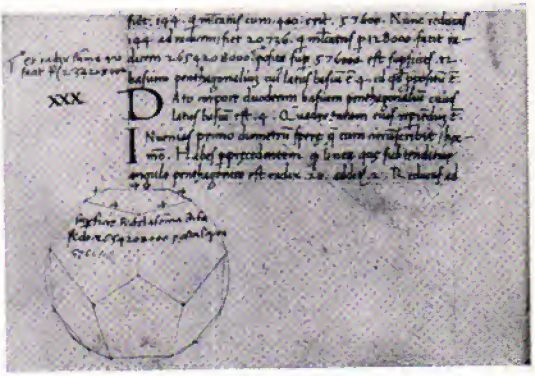


15. Portrait of Luca Pacioli. Jacopo de' Barbari, Museo di Capodimonte, Naples.

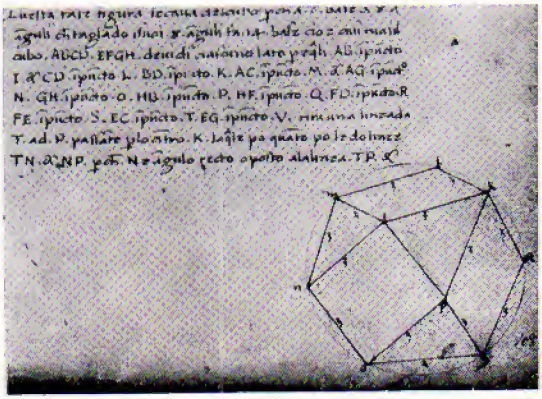
16. Portrait of Luca Pacioli. L. Pacioli, *Summa de arithmetica*, fol. 111v.

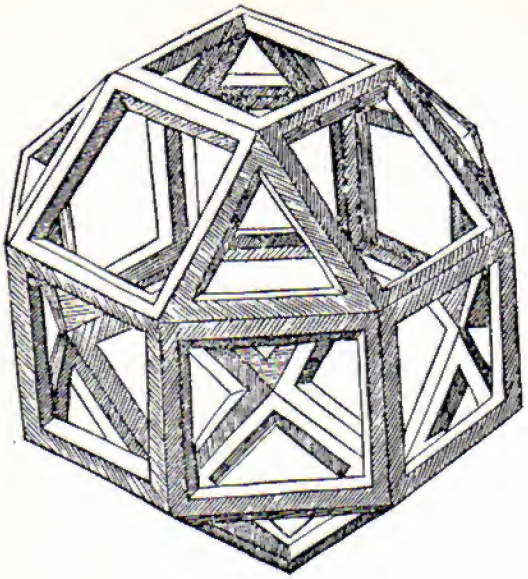
17. Icosahedron. Piero, *Trattato d'abaco*, fol. 113v.

18. Dodecahedron. Piero, *Libellus de quinque corporibus*, fol. 33v.



19. Irregular body. Piero, *Trattato d'abaco*, fol. 108r.





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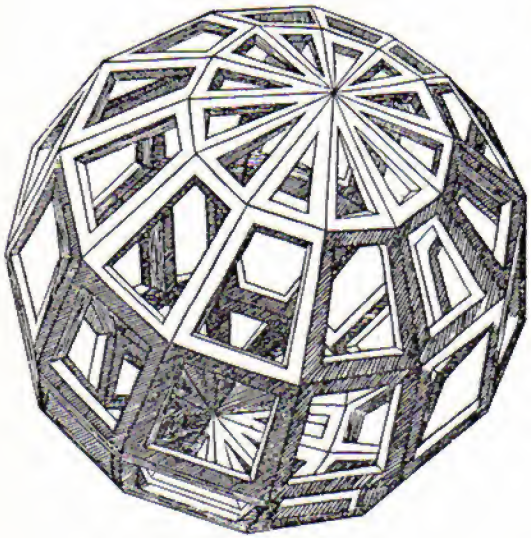

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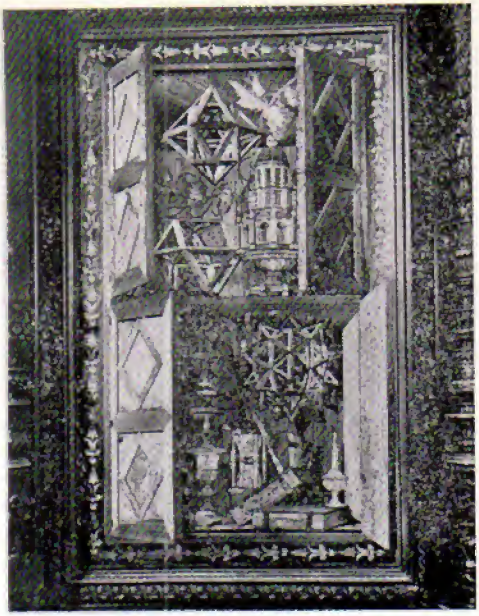

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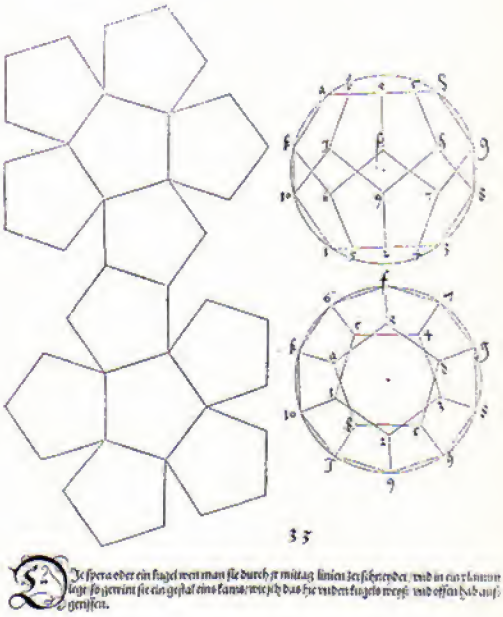

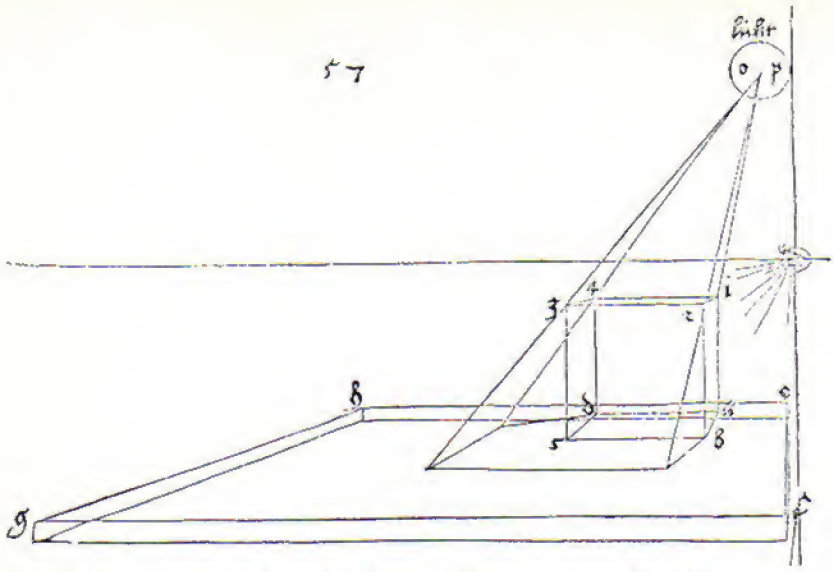

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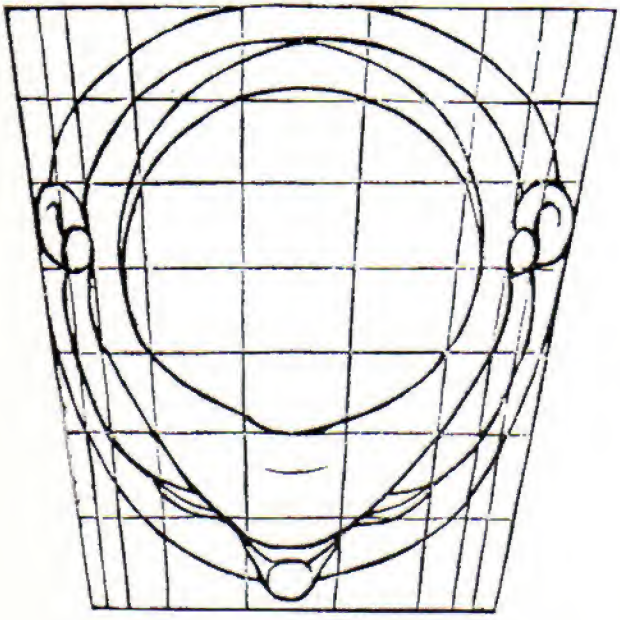

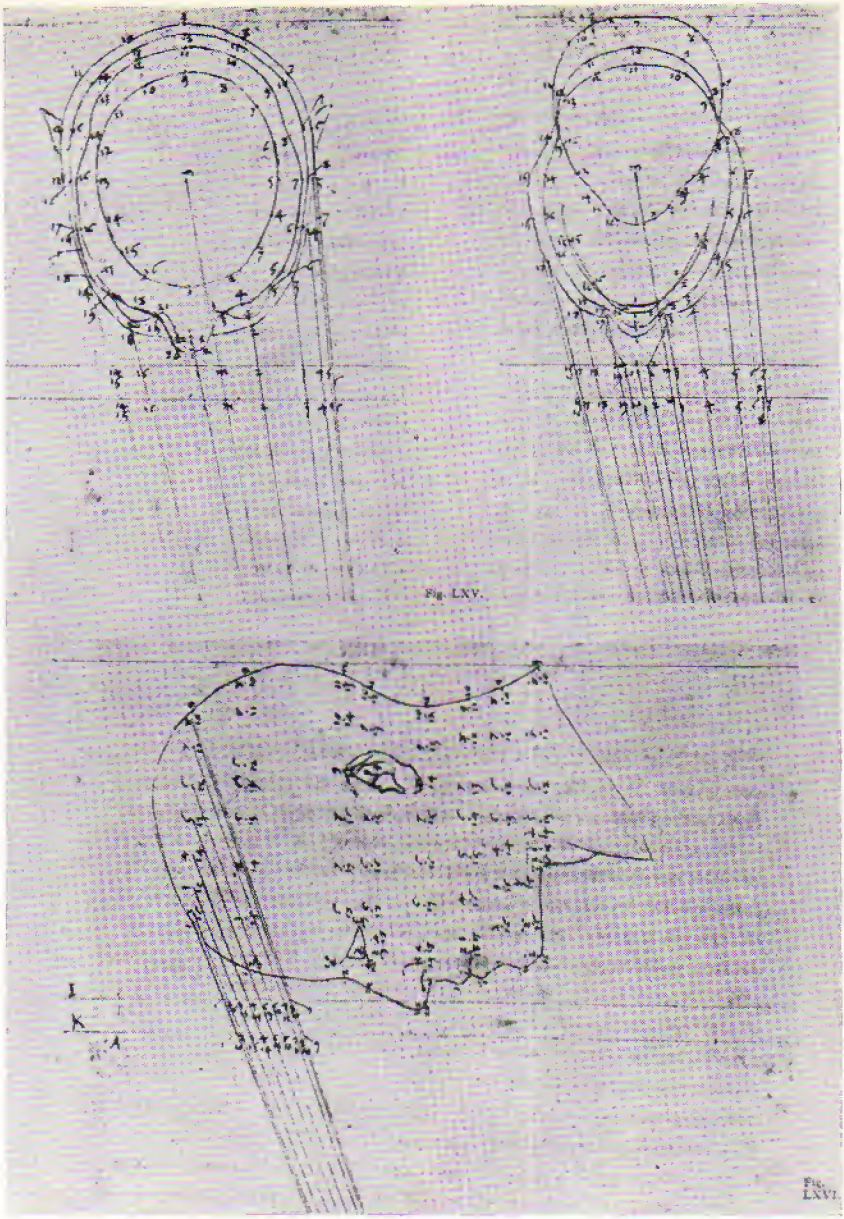

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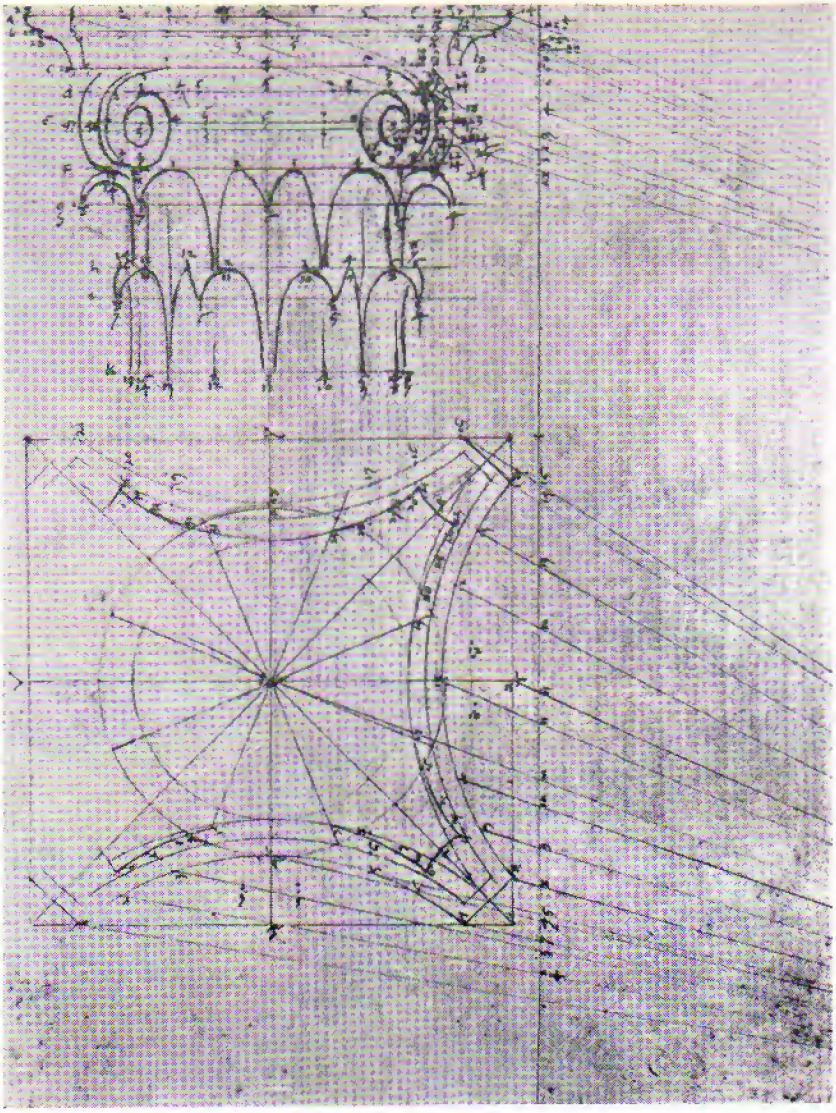

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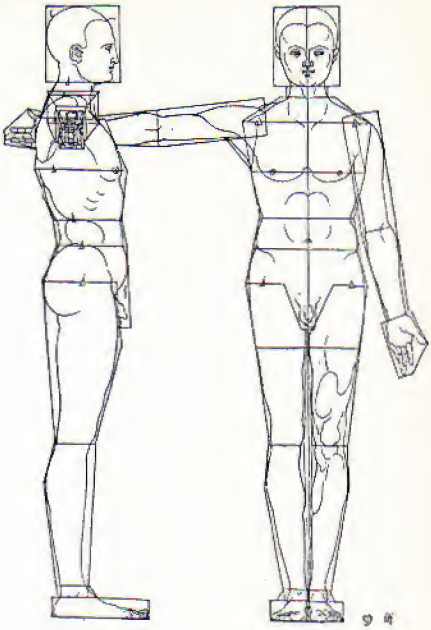

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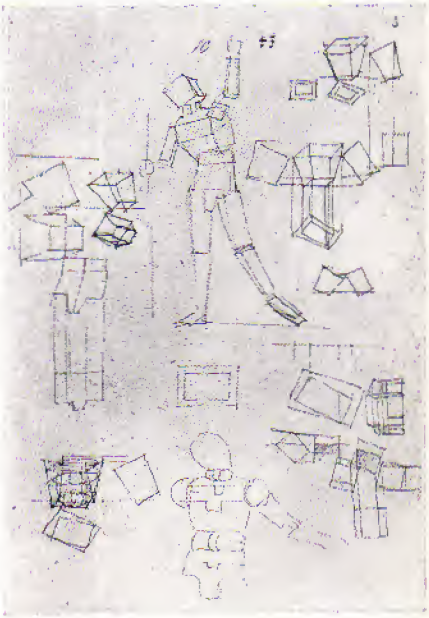

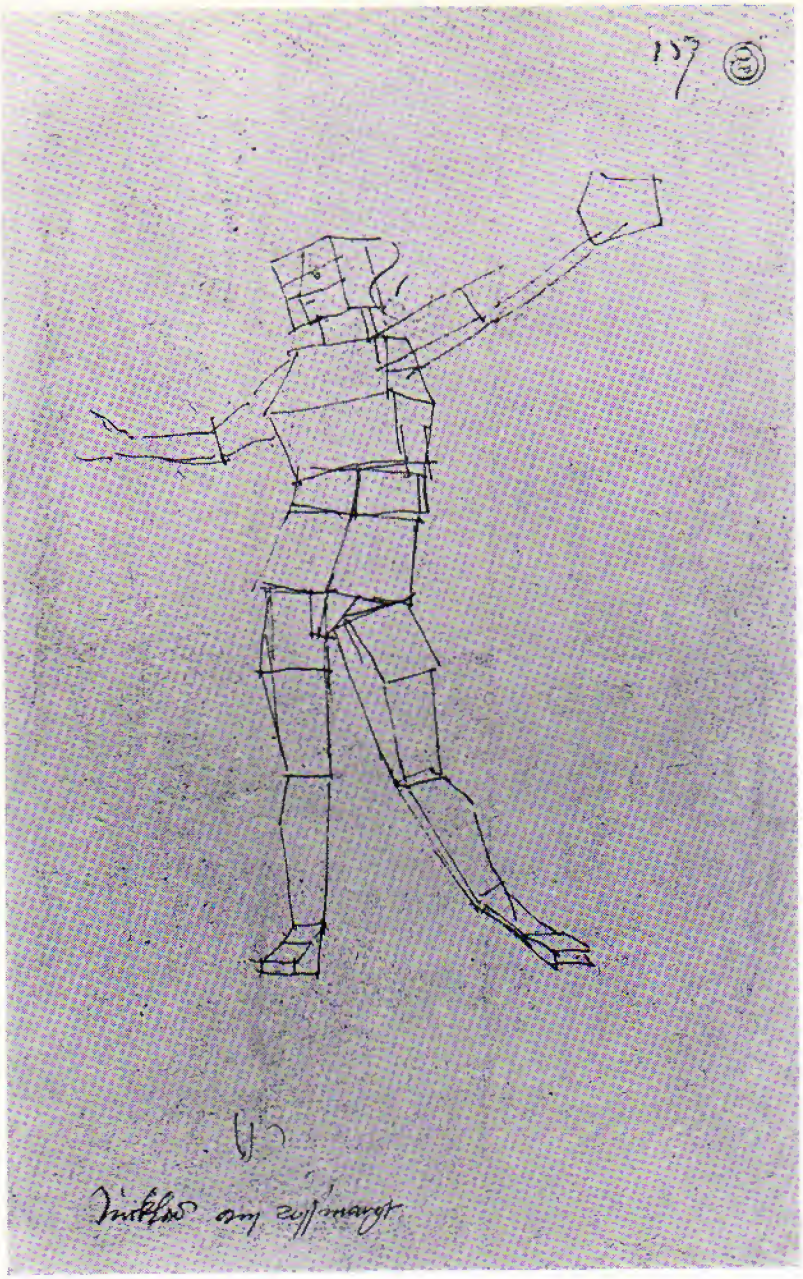

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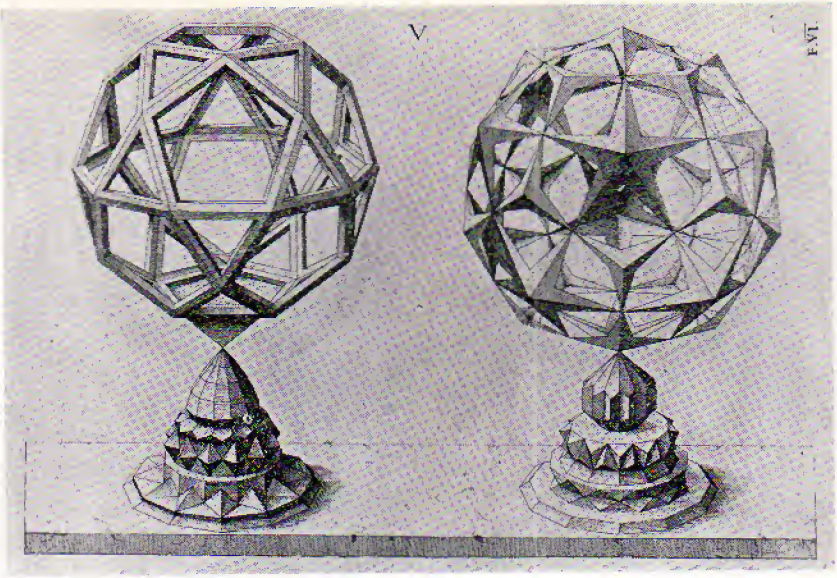

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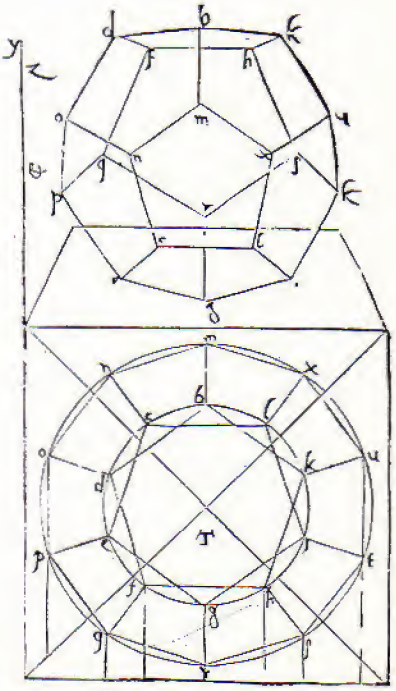

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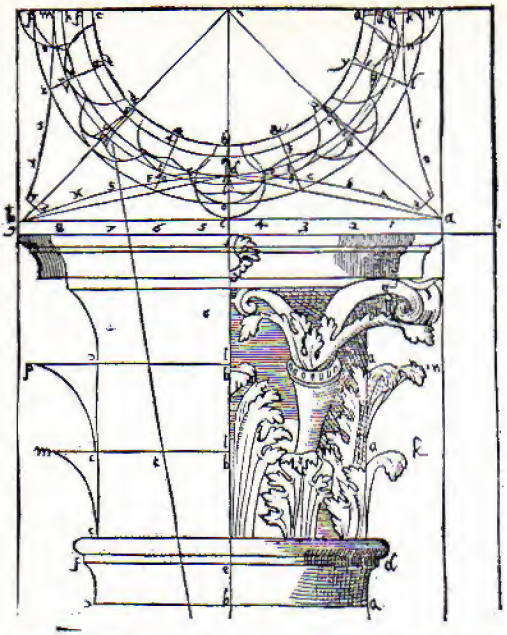

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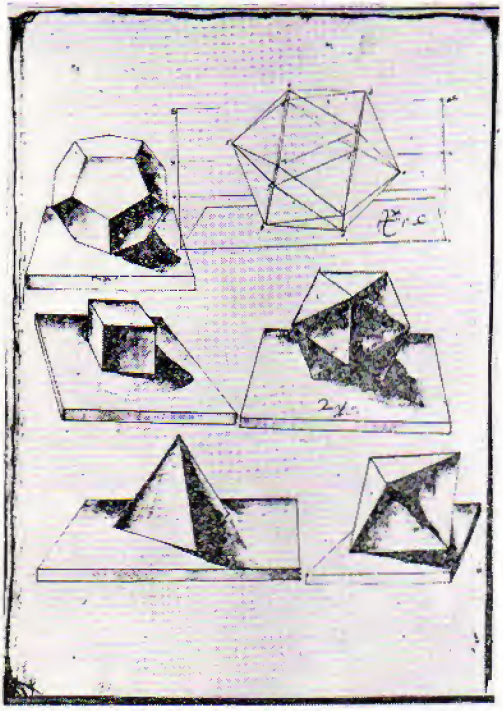

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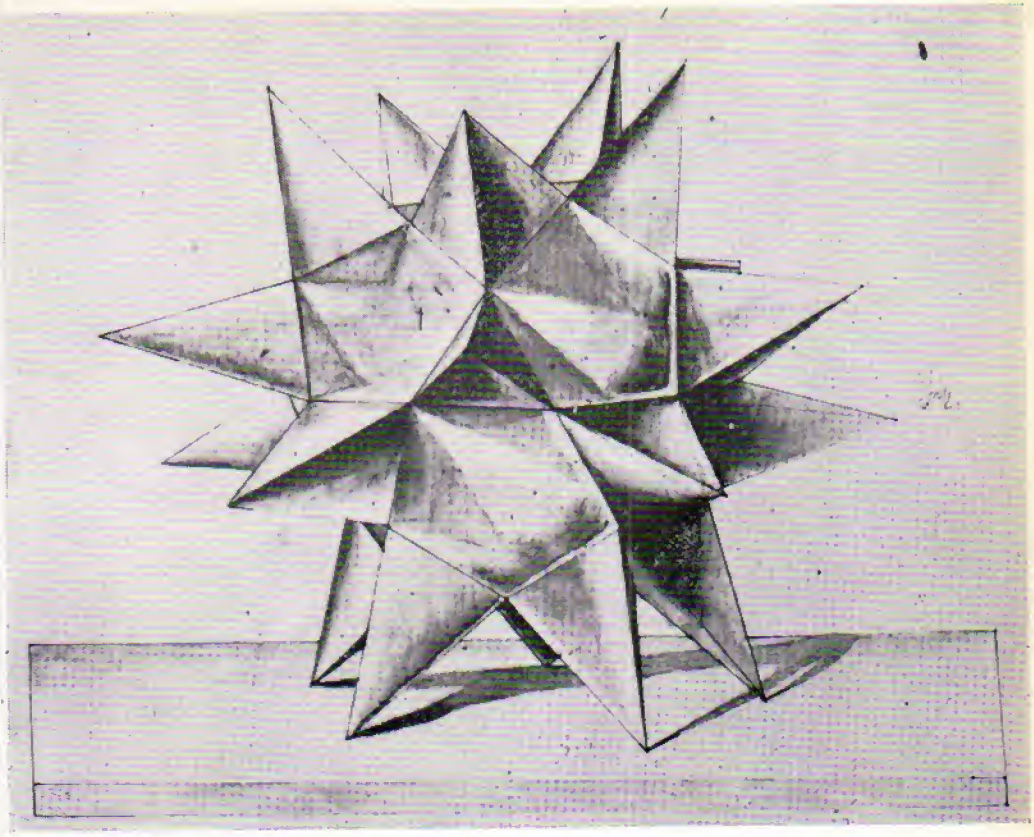

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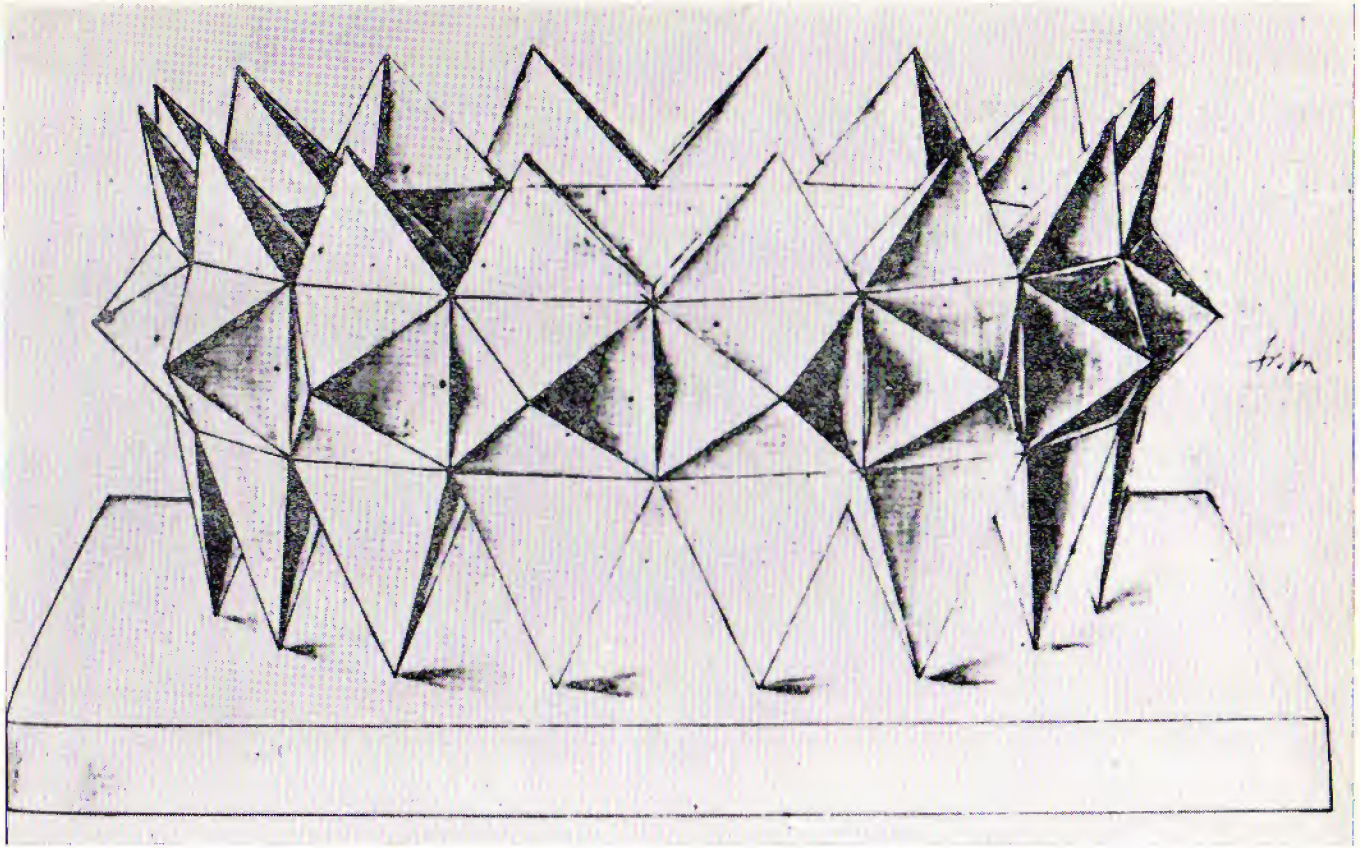

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This volume is a study in the relationship between Renaissance mathematical theory and artistic practice. The author examines the specific case of Piero, the artist and mathematician, whose paintings are today acclaimed above all others for their geometric perfection. Although his treatise on perspective is discussed whenever perspective is the order of the day, Piero's two purely mathematical works, the *Trattato d'aboco* and the *Libellus de quinque corporibus regularibus,* have been neglected by historians of art. It is, however, through these two « minor » treatise that Piero's mathematical abilities, highly celebrated in his own time, are brought once again into focus. The author investigates these works in close detail, tracing both their sources in classical and late-medieval mathematics and the internal development of Piero's mathematics that they reveal, as well as their relationships with Piero's perspective treatise. Following the exposition of the content of these works, the study analyzes the implications of Piero's ideas and their diffusion in the realm of theory and practice of the figurative arts and architecture.

MARGARET DALY DAVIS has studied the history of art in America and Europe. A graduate of New York University, she received the Ph. D. from the University of North Carolina at Chapel Hill. In recent years she has been living in Florence.

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